On Use of a New Method of Solution of Darboux Problem for Solution of the Problem of Motion of a Ball on a Routh Plane

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Abstract

The problem of motion of a homogeneous ball on a routh horizontal plane is considered. It is known that if the ball rolls with sliding then the angular velocity vector is a linear function if time: $\underset{\sim}{\omega}(t) = \underset{\sim}{a}t + \underset{\sim}{b}$. To solve the problem of rotational motion of the ball completely it is necessary to find the tensor of turn that is to solve Darboux problem $\dot{\underline{\underline{P}}} = \underset{\sim}{\omega} \times \underline{\underline{P}}$. In general case (when vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$ is not parallel) this problem has not been solved up to now and it will be solved below.

1 Introduction.

It is known that only some problems of the rigid body dynamics can be solved analytically. Now the methods of mechanics by Hamilton are usually used for investigation of problems of the rigid body dynamics. Unfortunately, the methods of Hamilton's mechanics can be applied to the conservative problems or the problems with small nonconservative forces and moments. The most of problems of the rigid body dynamics is nonconservative. To solve these problems alternative methods should be developed. A new approach to solution of the rigid body dynamics problems is based on using of the tensor of turn. The tensor of turn was introduced in consideration many years ago [1], but it was not used in mechanics. For the first time the tensor of turn was applied in mechanics by P. A. Zhilin [2], [3]. At the next years the tensor of turn was used for formulation of the fundamental laws of mechanics and creation of some continuum theories (nonlinear theory of shells, nonlinear theory of bars, the theory of Kelvin's medium). Using of the tensor of turn in the rigid body dynamics began from papers by P. A. Zhilin [4], [5]. Method based on using of the tensor of turn allowed to obtain some new results. In particular a new approach to solution of Darboux problem was proposed [6]. This approach allows to reduce Darboux problem to the 3-ed order linear differential equation. Applying

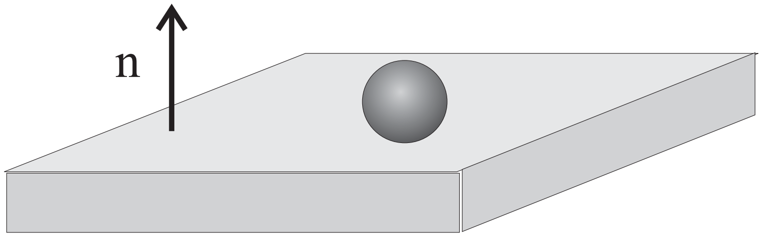

Figure 1: A ball on the plane

the method proposed in [6] to solution of the problem of motion of a homogeneous ball on the horizontal routh plane will be demonstrated below. This problem is well known. It was considered in many papers and books (for example [7]). However rotational motion oh the ball is not studied completely since in all known publications only angular velocity vector is found. Complete solution of this problem will be constructed below.

2 Formulation of problem. The known results.

Equations of motion of the ball (see figure 1) are

$$m\dot{\underline{V}}_C = \underline{F}_{fr} + m\underline{g} + \underline{N}\,, \qquad (\underline{\underline{\theta}} \cdot \underline{\omega})^{\cdot} = \underline{R}_{CA} \times (\underline{F}_{fr} + \underline{N}) \tag{1}$$

where the force of friction $\underline{F}_{fr}$ is

$$\underline{F}_{fr} = \begin{cases} -\mu\, N\; \underline{V}_A \,/\, |\underline{V}_A|\,, & \underline{V}_A \neq 0 \\ \underline{f}_{st}\,, & \underline{V}_A = 0 \end{cases} \tag{2}$$

Here m is the mass of the ball, $\underline{\underline{\theta}} = \theta\, \underline{\underline{E}}$ is the tensor of inertia of the ball, $\underline{\underline{E}}$ is the unit tensor, $\theta = \frac{2}{5}mr^2$, r is radius of the ball, $m\underline{g}$ is gravity, $\underline{N} = N\underline{n}$ is the normal reaction of the plane, $\underline{n}$ is the unit vector orthogonal to the plane, μ is the coefficient of sliding friction, $\underline{f}_{st}$ is the static force of friction, $\underline{V}_C$ is the center mass velocity, $\underline{V}_A$ is velocity of the lower point of the ball, $\underline{R}_{CA} = -r\underline{n}$. Taking into account relation $\underline{V}_A = \underline{V}_C + \underline{\omega} \times \underline{R}_{CA}$ the system (1), (2) can be rewritten as follows

$$\underline{V}_A \neq 0: \quad \dot{\underline{V}}_C = -\mu g\, \underline{V}_A \,/\, |\underline{V}_A|\,, \;\; \theta\, \dot{\underline{\omega}} = \mu m g r\, \underline{n} \times \underline{V}_A \,/\, |\underline{V}_A|\,, \;\; \underline{V}_A = \underline{V}_C + r\, \underline{n} \times \underline{\omega}$$

$$\underline{V}_A = 0: \quad m\dot{\underline{V}}_C = \underline{f}_{st}\,, \;\; \theta\, \dot{\underline{\omega}} = -r\, \underline{n} \times \underline{f}_{st}\,, \;\; \underline{V}_C + r\, \underline{n} \times \underline{\omega} = 0 \tag{3}$$

If the initial values of the angular velocity and the center mass velocity are related by equation $\underline{V}_0 + r\, \underline{n} \times \underline{\omega}_0 = 0$, then the ball will roll without sliding

$$\underline{V}_C(t) = \underline{V}_0\,, \qquad \underline{\omega}(t) = \underline{\omega}_0\,, \qquad \underline{f}_{st} = 0 \tag{4}$$

If the initial values of the angular velocity and the center mass velocity satisfy condition $\underline{V}_0 + r\, \underline{n} \times \underline{\omega}_0 \neq 0$, then the ball will roll with sliding till instant t_*. At the instant t_* velocity of the lower point of the ball $\underline{V}_A$ become equal to zero and the ball begin to roll without sliding.

$$\begin{aligned} 0 \le t \le t_* : \qquad & \underline{V}_C(t) = -\mu g\, t \underline{e} + \underline{V}_0 \,, \qquad \underline{\omega}(t) = \frac{\mu m g r}{\theta}\, t\, \underline{n} \times \underline{e} + \underline{\omega}_0 \\ t \ge t_* : \qquad & \underline{V}_C(t) = \underline{V}_C(t_*) \,, \qquad \underline{\omega}(t) = \underline{\omega}(t_*) \,, \qquad \underline{f}_{st} = 0 \end{aligned} \tag{5}$$

Here

$$t_* = \frac{|\underline{V}_0 + r\, \underline{n} \times \underline{\omega}_0|}{\mu g (1 + m r^2/\theta)} \,, \qquad \underline{e} = \frac{\underline{V}_0 + r\, \underline{n} \times \underline{\omega}_0}{|\underline{V}_0 + r\, \underline{n} \times \underline{\omega}_0|} \tag{6}$$

Equations (5) determine the motion of the mass center of the ball completely

$$\begin{aligned} 0 \le t \le t_* : \qquad & \underline{R}_C(t) = -\tfrac{\mu g}{2}\, t^2\, \underline{e} + \underline{V}_0\, t \\ t \ge t_* : \qquad & \underline{R}_C(t) = (-\mu g\, t_* \underline{e} + \underline{V}_0)\, t + \tfrac{\mu g}{2}\, t_*^2\, \underline{e} \end{aligned} \tag{7}$$

Rotational motion of the ball is not determined completely since solution of the problem of determination of the tensor of turn by the angular velocity vector (Darboux problem) is not known for arbitrary angular velocity vector. At the interval of time $0 \le t \le t_*$ the angular velocity vector is the linear function of time (5). Solution of the Darboux problem is known for particular case $\underline{n} \times \underline{e} \parallel \underline{\omega}_0$ only. The problem of determination of rotational motion of the ball in general case will be solved below.

3 The tensor of turn. Darboux problem.

Definition. The properly orthogonal tensor which is solution of equations

$$\underline{\underline{P}} \cdot \underline{\underline{P}}^T = \underline{\underline{P}}^T \cdot \underline{\underline{P}} = \underline{\underline{E}} \,, \qquad det\, \underline{\underline{P}} = +1 \tag{8}$$

is called tensor of turn.

Definition. Tensors $\underline{\underline{S}}(t)$ and $\underline{\underline{S}}_r(t)$, calculated by formulae

$$\underline{\underline{S}}(t) = \dot{\underline{\underline{P}}}(t) \cdot \underline{\underline{P}}^T(t) \,, \qquad \underline{\underline{S}}_r(t) = \underline{\underline{P}}^T(t) \cdot \dot{\underline{\underline{P}}}(t) \tag{9}$$

are called left spin tensor and right spin tensor accordingly.

Definition. Accompanying vector of the left spin tensor $\underline{\omega}(t)$ defining by formula

$$\underline{\underline{S}}(t) = \underline{\omega}(t) \times \underline{\underline{E}} \tag{10}$$

is called left angular velocity vector.

Definition. Accompanying vector of the right spin tensor $\underline{\Omega}(t)$ defining by formula

$$\underline{\underline{S}}_r(t) = \underline{\Omega}(t) \times \underline{\underline{E}} \tag{11}$$

is called right angular velocity vector.

The left Darboux problem. Let the left angular velocity vector $\underline{\omega}(t)$ be known. The tensor of turn should be found. This problem is formulated as follows

$$\dot{\underline{\underline{P}}}(t) = \underline{\omega}(t) \times \underline{\underline{P}}(t) \,, \qquad \underline{\underline{P}}(t_0) = \underline{\underline{P}}_0 \tag{12}$$

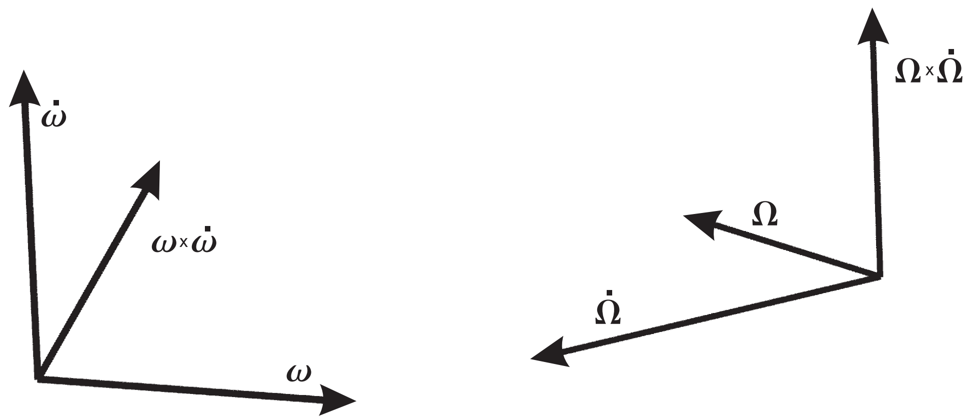

Figure 2: $\underline{\omega} = \underline{\underline{P}} \cdot \underline{\Omega}\,,\ \dot{\underline{\omega}} = \underline{\underline{P}} \cdot \dot{\underline{\Omega}}\,,\ \underline{\omega} \times \dot{\underline{\omega}} = \underline{\underline{P}} \cdot (\underline{\Omega} \times \dot{\underline{\Omega}})$

The right Darboux problem. Let the right angular velocity vector $\underline{\Omega}(t)$ be known. The tensor of turn should be found. This problem is formulated as follows

$$\dot{\underline{\underline{P}}}(t) = \underline{\underline{P}}(t) \times \underline{\Omega}(t)\,, \qquad \underline{\underline{P}}(0) = \underline{\underline{P}}_0 \tag{13}$$

The theorem of representation of the tensor of turn by the left and right angular velocity vectors. Let both angular velocity vectors be known. Then the tensor of turn can be found without additional integration. It is calculated by formula

$$\begin{gathered} \underline{\underline{P}} = A\,\underline{\omega}\,\underline{\Omega} + B\,(\underline{\omega}\,\underline{\Omega})^{\cdot} + C\,\dot{\underline{\omega}}\,\dot{\underline{\Omega}} + D\,(\underline{\omega} \times \dot{\underline{\omega}})(\underline{\Omega} \times \dot{\underline{\Omega}}) \\ A = \dot{\underline{\Omega}} \cdot \dot{\underline{\Omega}}\,D = \dot{\underline{\omega}} \cdot \dot{\underline{\omega}}\,D\,, \qquad B = -\tfrac{1}{2}(\Omega^2)^{\cdot}\,D = -\tfrac{1}{2}(\omega^2)^{\cdot}\,D \\ C = \Omega^2\,D = \omega^2\,D\,, \qquad D = 1/(\underline{\Omega} \times \dot{\underline{\Omega}})^2 = 1/(\underline{\omega} \times \dot{\underline{\omega}})^2 \end{gathered} \tag{14}$$

Proof of this theorem can be found in [6]. It is based on relations between the left and right angular velocity vectors: $\underline{\omega} = \underline{\underline{P}} \cdot \underline{\Omega}\,,\ \dot{\underline{\omega}} = \underline{\underline{P}} \cdot \dot{\underline{\Omega}}\,,\ \underline{\omega} \times \dot{\underline{\omega}} = \underline{\underline{P}} \cdot (\underline{\Omega} \times \dot{\underline{\Omega}})$ (see figure 2). Theorem (14) allows to formulate Darboux problem as follows.

The left Darboux problem. The left angular velocity vector $\underline{\omega}(t)$ is known. The right angular velocity vector $\underline{\Omega}(t)$ should be found.

The right Darboux problem. The right angular velocity vector $\underline{\Omega}(t)$ is known. The left angular velocity vector $\underline{\omega}(t)$ should be found.

Two formulations of the left Darboux problem were proposed in [6].

Formulation I. The right angular velocity vector is determined as solution of Cauchy problem

$$\begin{gathered} \ddot{\underline{\Omega}} + a(\underline{\omega})\,\dot{\underline{\Omega}} + b(\underline{\omega})\,\underline{\Omega} = c(\underline{\omega})\,\underline{\Omega} \times \dot{\underline{\Omega}}\,, \qquad \underline{\Omega}(t_0) = \underline{\underline{P}}_0^T \cdot \underline{\omega}_0\,, \quad \dot{\underline{\Omega}}(t_0) = \underline{\underline{P}}_0^T \cdot \dot{\underline{\omega}}_0 \\ a(\underline{\omega}) = -(\ln|\underline{\omega} \times \dot{\underline{\omega}}|)^{\cdot}, \quad b(\underline{\omega}) = \frac{(\underline{\omega} \times \dot{\underline{\omega}}) \cdot (\dot{\underline{\omega}} \times \ddot{\underline{\omega}})}{(\underline{\omega} \times \dot{\underline{\omega}})^2}\,, \quad c(\underline{\omega}) = \frac{\ddot{\underline{\omega}} \cdot (\underline{\omega} \times \dot{\underline{\omega}})}{(\underline{\omega} \times \dot{\underline{\omega}})^2} - 1 \end{gathered} \tag{15}$$

Formulation II. The right angular velocity vector is determined as solution of

Cauchy problem

$$\dddot{\underline{\Omega}} + A(\underline{\omega})\,\ddot{\underline{\Omega}} + B(\underline{\omega})\,\dot{\underline{\Omega}} + C(\underline{\omega})\,\underline{\Omega} = 0$$

$$\underline{\Omega}(t_0) = \underline{\underline{P}}_0^T \cdot \underline{\omega}_0\,, \qquad \dot{\underline{\Omega}}(t_0) = \underline{\underline{P}}_0^T \cdot \dot{\underline{\omega}}_0\,,$$

$$\ddot{\underline{\Omega}}(t_0) = \underline{\underline{P}}_0^T \cdot [c(\underline{\omega}_0)\,\underline{\omega}_0 \times \dot{\underline{\omega}}_0 - a(\underline{\omega}_0)\,\dot{\underline{\omega}}_0 - b(\underline{\omega}_0)\,\underline{\omega}_0]$$

$$A(\underline{\omega}) = 2a(\underline{\omega}) - [\ln c(\underline{\omega})]^{\cdot}, \quad B(\underline{\omega}) = \dot{a}(\underline{\omega}) + a(\underline{\omega})\,(a(\underline{\omega}) - [\ln c(\underline{\omega})]^{\cdot}) + b(\underline{\omega}) + c^2(\underline{\omega})\,\omega^2$$

$$C(\underline{\omega}) = \dot{b}(\underline{\omega}) + b(\underline{\omega})\,(a(\underline{\omega}) - [\ln c(\underline{\omega})]^{\cdot}) - \tfrac{1}{2}c^2(\underline{\omega})\,(\omega^2)^{\cdot} \tag{16}$$

Two formulations of the right Darboux problem are similar (15), (16).

4 Solution of Darboux problem in the case $\underline{\omega}(t)$ is a linear function of time.

According to formula (5) the problem of rolling with sliding of a homogeneous ball on the horizontal plan can be reduced to the problem of determination of the tensor of turn corresponding the left angular velocity vector

$$\underline{\omega}(t) = \underline{a}\,t + \underline{b}\,, \qquad \underline{a} = \frac{\mu m g r\,\underline{n} \times (\underline{V}_0 + r\,\underline{n} \times \underline{\omega}_0)}{\theta\,|\underline{V}_0 + r\,\underline{n} \times \underline{\omega}_0|}\,, \qquad \underline{b} = \underline{\omega}_0 \tag{17}$$

According to formula (14) the tensor of turn corresponding the left angular velocity vector $\underline{\omega}(t) = \underline{a}\,t + \underline{b}$ can be calculated by formula

$$\underline{\underline{P}}(t) = \frac{\underline{a}}{a^2}\,\dot{\underline{\Omega}}(t) + \frac{\underline{a}t_0 + \underline{b}}{2a\text{æ}}\left(\underline{\Omega}(t) - (t-t_0)\dot{\underline{\Omega}}(t)\right) - \frac{\underline{a}\times\underline{b}}{2a^3\text{æ}}\left(\underline{\Omega}(t)\times\dot{\underline{\Omega}}(t)\right)$$
$$t_0 = -\frac{\underline{a}\cdot\underline{b}}{a^2}\,, \qquad \text{æ} = \frac{a^2\,b^2 - (\underline{a}\cdot\underline{b})^2}{2\,a^3}\,, \qquad a = |\underline{a}|\,, \quad b = |\underline{b}| \tag{18}$$

where the right angular velocity vector $\underline{\Omega}(t)$ is determined solution of Cauchy problem (15) or Cauchy problem (16). In the case $\underline{\omega}(t) = \underline{a}\,t + \underline{b}$ Cauchy problem (15) takes a form

$$\ddot{\underline{\Omega}}(t) = -\underline{\Omega}(t)\times\dot{\underline{\Omega}}(t)\,, \qquad \underline{\Omega}(t_0) = \underline{a}\,t_0 + \underline{b}\,, \qquad \dot{\underline{\Omega}}(t_0) = \underline{a} \qquad \left(\underline{\underline{P}}(t_0) = \underline{\underline{E}}\right) \tag{19}$$

and Cauchy problem (16) takes a form

$$\dddot{\underline{\Omega}}(t) + [a^2(t-t_0)^2 + 2a\text{æ}]\;\dot{\underline{\Omega}}(t) - a^2(t-t_0)\,\underline{\Omega}(t) = 0$$
$$\underline{\Omega}(t_0) = \underline{a}\,t_0 + \underline{b}\,, \qquad \dot{\underline{\Omega}}(t_0) = \underline{a}\,, \qquad \ddot{\underline{\Omega}}(t_0) = \underline{a}\times\underline{b} \tag{20}$$

Let us represent vector $\underline{\Omega}(t)$ using the cylindrical coordinate system

$$\underline{\Omega}(t) = z(t)\,\underline{k} + r(t)\,\underline{e}_r[\varphi(t)]\,, \qquad \underline{e}_r[\varphi(t)] = \cos\varphi(t)\,\underline{i} + \sin\varphi(t)\,\underline{j}$$
$$\underline{k} = \frac{1}{a}\,\underline{a}\,, \qquad \underline{i} = \frac{1}{\sqrt{2a\text{æ}}}\,(\underline{a}\,t_0 + \underline{b})\,, \qquad \underline{j} = \frac{1}{\sqrt{2a^3\text{æ}}}\,\underline{a}\times\underline{b} \tag{21}$$

Let us substitute (21) in equations (20) and project the equations on vector $\underline{k}$. We obtain Cauchy problem in scalar function $z(t)$

$$\dddot{z}(t) + [a^2(t-t_0)^2 + 2a\text{æ}]\,\dot{z}(t) - a^2(t-t_0)\,z(t) = 0$$
$$z(t_0) = 0\,, \qquad \dot{z}(t_0) = a\,, \qquad \ddot{z}(t_0) = 0 \tag{22}$$

Let us introduce a new variable

$$\tau(t) = \frac{a}{2}(t-t_0)^2 \tag{23}$$

As a result differential equation (22) takes a form

$$\tau\, z'''(\tau) + \frac{3}{2}\, z''(\tau) + (\tau + \text{æ})\, z'(\tau) - \frac{1}{2}\, z(\tau) = 0 \tag{24}$$

Let us look for solution of equation (24) in a form

$$z(\tau) = C\frac{\sqrt{\tau}}{\text{æ}} - \int\limits_0^\tau \zeta(s)\sqrt{\tau - s}\, ds \tag{25}$$

where C is a constant and $\zeta(s)$ is unknown function. Substituting expression (25) in equation (24) we obtain

$$\frac{C - \zeta(0)}{\sqrt{\tau}} - \int\limits_0^\tau \Big[s\,\zeta''(s) + 2\,\zeta'(s) + (s+\text{æ})\,\zeta(s)\Big]\frac{1}{\sqrt{\tau - s}}\, ds = 0 \tag{26}$$

Equation (26) is true if the integrand expression and term outside the integral equal to zero

$$s\,\zeta''(s) + 2\,\zeta'(s) + (s+\text{æ})\,\zeta(s) = 0\,, \qquad \zeta(0) = C \tag{27}$$

Solution of differential equation (27) is known [8].

$$\zeta(s) = C\,\frac{\operatorname{Re}\Phi(1 - \frac{i\text{æ}}{2},\, 2,\, -2is)}{\cos s} \tag{28}$$

Here $\Phi(1 - \frac{i\text{æ}}{2},\, 2,\, -2is)$ is the degenerate hypergeometric function

$$\Phi(1 - \frac{i\text{æ}}{2},\, 2,\, -2is) = 1 + \sum_{k=1}^{\infty} \frac{(1-\frac{i\text{æ}}{2})(2-\frac{i\text{æ}}{2})...(k-\frac{i\text{æ}}{2})}{k!\,(k+1)!}\,(-2is)^k \tag{29}$$

Function $z(t)$ defined by expressions (23, (25), (28) is a particular solution of differential equation (22). It is easy to see that function $z(t)$ satisfies the first and the third initial conditions (22). The second initial condition (22) satisfies if

$$C = \sqrt{2a}\,\text{æ}\,\operatorname{sign}(t-t_0) \tag{30}$$

Thus solution of Cauchy problem (22) is

$$z(t) = a\,(t-t_0) - \sqrt{2a}\,\text{æ}\,\operatorname{sign}(t-t_0)\int\limits_0^{\frac{a}{2}(t-t_0)^2} \frac{\operatorname{Re}\Phi(1-\frac{i\text{æ}}{2}, 2, -2is)}{\cos s}\sqrt{\frac{a}{2}(t-t_0)^2 - s}\; ds \tag{31}$$

Relation $|\underline{\Omega}(t)| = |\underline{\omega}(t)|$ allows to express function $r(t)$ in terms $z(t)$

$$r(t) = \sqrt{a^2(t-t_0)^2 + 2a\text{æ} - z^2(t)} \tag{32}$$

Projecting equation (19) on vector $\underline{k}$ expression for angle $\varphi(t)$ is obtained

$$\varphi(t) = -\int \frac{\ddot{z}(t)}{a^2(t-t_0)^2 + 2a\text{æ} - z^2(t)}\, dt\,, \qquad \varphi(t_0) = 0 \tag{33}$$

Formulae (18), (21), (31), (32), (33) give the complete solution of Darboux problem in the case $\underline{\omega}(t) = \underline{a}\, t + \underline{b}$.

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