Lecture 2

Elena Ivanova

Modelling of continua with microstructure: equations of motion, strain measures and constitutive equations. Continua with inner rotational degrees of freedom.

1 Euler's mechanics

1.1 Body of a general kind

Let us consider a collection of body-points $\mathcal{A}_i$, which we call a body $\mathcal{A}$ (see Fig. 1). All remaining body-points are called the environment of body $\mathcal{A}$ and denoted by a symbol $\mathcal{A}^e$.

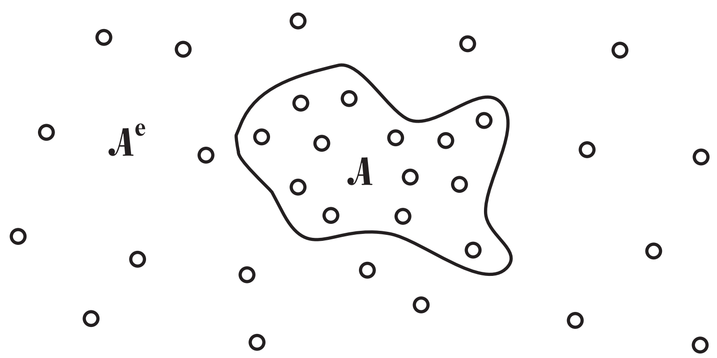

Figure 1: Body $\mathcal{A}$ and its environment $\mathcal{A}^e$

To model the action of the environment $\mathcal{A}^e$ on the body $\mathcal{A}$ we should assign a pair of vectors: a force vector and a moment vector. The force and moment vectors are additive on both the bodies compound the body $\mathcal{A}$ and the bodies compound its environment $\mathcal{A}^e$.

1.2 Interactions in a system of body-points

In Newton's mechanics (mechanics of point masses which possessing only translational degrees of freedom) interactions are characterized by forces.

For description of interactions of the body-points possessing not only translational but also rotational degrees of freedom the concept of force is insufficient. It is necessary to introduce the concept the properly moment as independent characteristic which can not be expressed in terms of forces.

Below we enunciate the concept of interactions of bodies of the general form, as suggested in works by P. A. Zhilin.

We denote the force acting on body A from body B by vector $\mathbf{F}(\mathcal{A},\mathcal{B})$. Thus force $\mathbf{F}(\mathcal{A},\mathcal{B})$ is reaction of body $\mathcal{B}$ on change of a spatial position of body $\mathcal{A}$.

Definition: Moment $\mathbf{M}^Q(\mathcal{A},\mathcal{B})$ acting on body $\mathcal{A}$ from body $\mathcal{B}$, which is calculated with respect to a reference point Q, can be expressed as follows

$$\mathbf{M}^Q(\mathcal{A},\mathcal{B}) = (\mathbf{R}_P - \mathbf{R}_Q) \times \mathbf{F}(\mathcal{A},\mathcal{B}) + \mathbf{L}^P(\mathcal{A},\mathcal{B}), \tag{1}$$

where vector $\mathbf{R}_Q$ defines the position of a reference point Q, vector $\mathbf{R}_P$ defines the position of a datum point P. The reference point Q can be chosen arbitrary but it should be fixed (motionless). The first term on the right-hand side of Eq. (1) is called the moment of force. Vector $\mathbf{L}^P(\mathcal{A},\mathcal{B})$ is called the proper moment. It depends on the choice of a datum point P but not on the choice of a reference point Q. The proper moment $\mathbf{L}^P(\mathcal{A},\mathcal{B})$ is reaction of body $\mathcal{B}$ on rotation of body $\mathcal{A}$ about the datum point P.

By definition, the full moment $\mathbf{M}^Q(\mathcal{A},\mathcal{B})$ does not depend on the choice of a datum point. In other words, the datum point being changed, the properly moment vector varies in such a way that the full moment vector $\mathbf{M}^Q(\mathcal{A},\mathcal{B})$ remains unchanged. Consequently,

$$\mathbf{L}^S(\mathcal{A},\mathcal{B}) = (\mathbf{R}_P - \mathbf{R}_S) \times \mathbf{F}(\mathcal{A},\mathcal{B}) + \mathbf{L}^P(\mathcal{A},\mathcal{B}). \tag{2}$$

Eq. (2) provides us with the relation between the proper moments calculated with respect to different datum points.

1.3 The balance equations in Euler's mechanics

In Newton's mechanics the equations of momentum balance, angular momentum balance and energy balance for system of point masses follow from the second Newton's law. In Euler's mechanics that considers the motion of the particles possessing rotational degrees of freedom and an internal structure all the balance equations are independent laws. Here we briefly formulate two fundamental laws of Euler's mechanics. The equation of energy balance will be formulated further.

The equation of momentum balance. The rate in the momentum change of body $\mathcal{A}$ is equal to the force acting on the body $\mathcal{A}$ from its environment plus the rate of the momentum supply in body $\mathcal{A}$, namely:

$$\frac{d\mathbf{K}_1(\mathcal{A})}{dt} = \mathbf{F}(\mathcal{A},\mathcal{A}^e) + \mathbf{k}_1(\mathcal{A}). \tag{3}$$

Here $\mathbf{F}(\mathcal{A},\mathcal{A}^e)$ is the force acting on the body $\mathcal{A}$ from its environment $\mathcal{A}^e$, and $\mathbf{k}_1(\mathcal{A})$ is the rate of the momentum supply in body $\mathcal{A}$.

The equation of angular momentum balance. The rate in the angular momentum change of body $\mathcal{A}$, calculated with respect to a reference point Q, is equal to the moment acting on body $\mathcal{A}$ from its environment, calculated with respect to the same reference point Q, plus the rate of the angular momentum supply in a body $\mathcal{A}$, namely:

$$\frac{d\mathbf{K}_2^Q(\mathcal{A})}{dt} = \mathbf{M}^Q(\mathcal{A}, \mathcal{A}^e) + \mathbf{k}_2^Q(\mathcal{A}). \tag{4}$$

Here $\mathbf{M}^Q(\mathcal{A}, \mathcal{A}^e)$ is the moment acting on the body $\mathcal{A}$ from its environment $\mathcal{A}^e$, and $\mathbf{k}_2^Q(\mathcal{A})$ is the rate of the angular momentum supply in body $\mathcal{A}$.

1.4 The equation of energy balance

The equation of energy balance is the third fundamental law of mechanics. Often the equation of energy balance is called the first law of thermodynamics or the first principle of thermodynamics.

The energy balance equation. The rate in the total energy change of body $\mathcal{A}$ is equal to the external force and moment power $N(\mathcal{A}, \mathcal{A}^e)$ plus the rate of supply of the energy of non-mechanical nature $\varepsilon(\mathcal{A})$:

$$\frac{dE(\mathcal{A})}{dt} = N(\mathcal{A}, \mathcal{A}^e) + \varepsilon(\mathcal{A}). \tag{5}$$

Here the total energy of a body $E(\mathcal{A})$ is a sum of the kinetic energy $K(\mathcal{A})$ and the internal energy $U(\mathcal{A})$. The power of external actions on body $\mathcal{A}$ consisting of body-points $\mathcal{A}_i$ is the bilinear form of velocities and actions:

$$N(\mathcal{A}, \mathcal{A}^e) = \sum_i \Big[\mathbf{F}(\mathcal{A}_i, \mathcal{A}^e) \cdot \mathbf{v}_i + \mathbf{L}(\mathcal{A}_i, \mathcal{A}^e) \cdot \boldsymbol{\omega}_i\Big]. \tag{6}$$

It is important to notice that the power of external actions depends on the proper moments $\mathbf{L}(\mathcal{A}_i, \mathcal{A}^e)$ rather than the full moments $\mathbf{M}^Q(\mathcal{A}_i, \mathcal{A}^e)$.

2 Continuum consisting of one-rotor gyrostats

2.1 Kinematics of the continuum

The one-rotor gyrostat (see Fig. 2) is a complex object which consists of the carrier body and the rotor. The rotor can rotate independently of rotation of the carrier body, but it can not translate relative to the carrier body. The carrier bodies of the gyrostats are considered to be the infinitesimal rigid bodies. The rotors of the gyrostats represent body-points whose tensors of inertia are the spherical parts of tensors. The kinetic energy of such body-point takes the form

$$K = m_* \left(\frac{1}{2}\mathbf{v} \cdot \mathbf{v} + B\,\mathbf{v} \cdot \boldsymbol{\omega} + \frac{1}{2}J\,\boldsymbol{\omega} \cdot \boldsymbol{\omega}\right). \tag{7}$$

Here m_* is the mass of the body-point, B and J are the moments of inertia of the body-point.

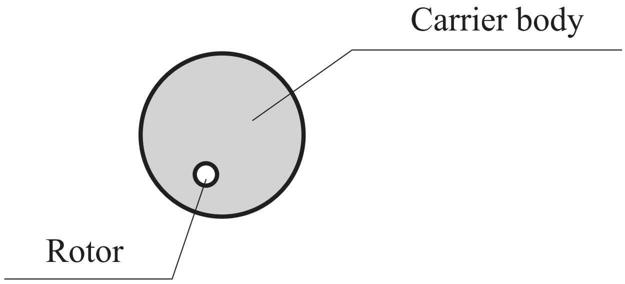

Figure 2: One-rotor gyrostat

The material medium (see Fig. 3) consisting of one-rotor gyrostats is considered. To derive the dynamic equations of the continuum we apply the spatial description. Let vector $\mathbf{r}$ determine the position of some point of space. We introduce following notations: $\mathbf{v}(\mathbf{r},t)$ is the velocity field; $\mathbf{u}(\mathbf{r},t)$ is the displacement field; $\tilde{\mathbf{P}}(\mathbf{r},t)$ and $\tilde{\boldsymbol{\omega}}(\mathbf{r},t)$ are the fields of the rotation tensors and the angular velocity vectors of the carrier bodies; $\mathbf{P}(\mathbf{r},t)$, $\boldsymbol{\omega}(\mathbf{r},t)$ are fields of the rotation tensors and the angular velocity vectors of the rotors.

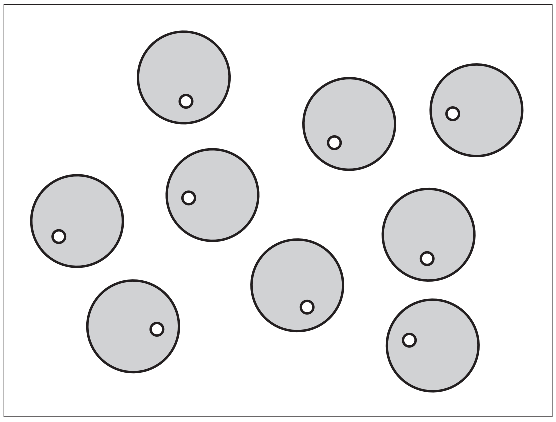

Figure 3: Elementary volume of continuum consisting of one-rotor gyrostats

In the spatial description the formulas relating the velocity vector to the displacement vector and also the angular velocity vectors to the rotation tensors are written down by means of the material derivative:

$$\mathbf{v} = \frac{\delta \mathbf{u}}{\delta t}, \qquad \tilde{\boldsymbol{\omega}} = -\frac{1}{2}\left(\frac{\delta \tilde{\mathbf{P}}}{\delta t}\cdot \tilde{\mathbf{P}}^T\right)_\times, \qquad \boldsymbol{\omega} = -\frac{1}{2}\left(\frac{\delta \mathbf{P}}{\delta t}\cdot \mathbf{P}^T\right)_\times. \tag{8}$$

Here the material derivative $\dfrac{\delta}{\delta t}$ is defined as follows:

$$\frac{\delta}{\delta t}\mathbf{u}(\mathbf{r},t) = \lim_{\triangle t\to 0}\frac{\mathbf{u}(\mathbf{r}+\mathbf{v}\triangle t,\, t+\triangle t) - \mathbf{u}(\mathbf{r},t)}{\triangle t} \equiv \\ \equiv \frac{d}{dt}\mathbf{u}(\mathbf{r},t) + \mathbf{v}(\mathbf{r},t)\cdot\nabla\mathbf{u}(\mathbf{r},t). \tag{9}$$

2.2 Dynamical structure of the continuum

The inertia properties of continuum under consideration are characterized by the mass density of the material medium $\rho(\mathbf{r}, t)$ at a given point of space, the inertia tensor $\mathbf{I}_*$ of carrier bodies at the actual position, as well as the inertia moments B and J of rotors.

The particles of continuum under consideration possess the internal degrees of freedom. Therefore, in order to describe the motion of this continuum it is not sufficient to formulate the balance equations of the momentum and the angular momentum for the control volume of the continuum. It is necessary to add these equations to the balance equation of the angular momentum for the rotors in control volume of the continuum. Therefore below we need the densities of the momentum and the angular momentum of the carrier bodies

$$\rho\mathbf{K}_1^{(cb)} = \rho(1-\epsilon)\,\mathbf{v}, \qquad \rho\mathbf{K}_2^{(cb)} = \rho\Big[\mathbf{r}\times(1-\epsilon)\,\mathbf{v} + \mathbf{I}_*\cdot\tilde{\boldsymbol{\omega}}\Big], \tag{10}$$

and the momentum and the angular momentum of the rotors

$$\rho\mathbf{K}_1^{(rot)} = \rho(\epsilon\,\mathbf{v} + B\,\boldsymbol{\omega}), \qquad \rho\mathbf{K}_2^{(rot)} = \rho\Big[\mathbf{r}\times(\epsilon\,\mathbf{v} + B\,\boldsymbol{\omega}) + B\,\mathbf{v} + J\,\boldsymbol{\omega}\Big], \tag{11}$$

where the angular momentum densities are calculated with respect to the origin of the reference frame.

Dimensionless parameter ϵ in Eqs. (10), (11) characterizes the distribution of mass in the gyrostat: if m is the mass of the gyrostat then $(1-\epsilon)m$ is the mass of its carrier body and ϵm is the mass of its rotor. Below we will see that the value of parameter ϵ is not important.

The densities of momentum and the angular momentum of continuum under consideration are

$$\rho\mathbf{K}_1 = \rho\mathbf{K}_1^{(cb)} + \rho\mathbf{K}_1^{(rot)}, \qquad \rho\mathbf{K}_2 = \rho\mathbf{K}_2^{(cb)} + \rho\mathbf{K}_2^{(rot)}. \tag{12}$$

From Eqs. (10)–(12) it follows:

$$\rho\mathbf{K}_1 = \rho\big(\mathbf{v} + B\,\boldsymbol{\omega}\big), \qquad \rho\mathbf{K}_2 = \rho\Big[\mathbf{r}\times(\mathbf{v} + B\,\boldsymbol{\omega}) + \mathbf{I}_*\cdot\tilde{\boldsymbol{\omega}} + B\,\mathbf{v} + J\,\boldsymbol{\omega}\Big], \tag{13}$$

Below we need the density of the kinetic energy of continuum under consideration. It has the form

$$\rho K = \rho\left(\frac{1}{2}\,\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}\,\tilde{\boldsymbol{\omega}}\cdot\mathbf{I}_*\cdot\tilde{\boldsymbol{\omega}} + B\,\mathbf{v}\cdot\boldsymbol{\omega} + \frac{1}{2}J\,\boldsymbol{\omega}\cdot\boldsymbol{\omega}\right). \tag{14}$$

2.3 Law of mass conservation

Let V and S denote some fixed region in the reference frame (control volume) and its surface, respectively. Let us formulate the law of mass conservation for the control volume:

$$\frac{d}{dt}\int_{(V)} \rho(\mathbf{r}, t)\, dV = -\int_{(S)} \mathbf{n}\cdot\mathbf{v}(\mathbf{r}, t)\rho(\mathbf{r}, t)\, dS. \tag{15}$$

Here $\mathbf{n}$ denotes the unit vector of a normal to the surface S. Using standard line of reasoning we derive from the Eq. (15) the law of mass conservation in the local form

$$\frac{\delta\rho}{\delta t} + \rho\,\nabla\cdot\mathbf{v} = 0. \tag{16}$$

2.4 Momentum balance equations

Now we formulate the equation of balance of momentum (3) for the carrier bodies in control volume V:

$$\frac{d}{dt}\int_{(V)} \rho\mathbf{K}_1^{(cb)}dV = \int_{(V)} \rho(\mathbf{f}+\mathbf{F})dV + \int_{(S)} \boldsymbol{\tau}_n dS - \int_{(S)} (\mathbf{n}\cdot\mathbf{v})\rho\mathbf{K}_1^{(cb)}dS. \tag{17}$$

Here $\mathbf{f}$ is the mass density of external forces; $\mathbf{F}$ is the mass density of forces modelling the effect of the rotors; $\boldsymbol{\tau}_n$ is the force vector modelling the influence of surrounding medium on the carrier bodies of gyrostats being on surface S of control volume V. Next we formulate the equation of balance of momentum (3) for the rotors in control volume V:

$$\frac{d}{dt}\int_{(V)} \rho\mathbf{K}_1^{(rot)}dV = -\int_{(V)} \rho\mathbf{F}dV - \int_{(S)} (\mathbf{n}\cdot\mathbf{v})\rho\mathbf{K}_1^{(rot)}dS. \tag{18}$$

The rotors are assumed to interact only by moment rather than forces. External force actions on the rotors is also supposed to be absent.

By standard reasoning we introduce the concept of stress tensor concerned with the force vector $\boldsymbol{\tau}_n$ by relation $\boldsymbol{\tau}_n = \mathbf{n}\cdot\boldsymbol{\tau}$ and from Eqs. (17), (18) we derive the local form of the momentum balance equations for the carrier bodies and the rotors:

$$\nabla\cdot\boldsymbol{\tau} + \rho(\mathbf{f}+\mathbf{F}) = \rho(1-\epsilon)\,\frac{\delta\mathbf{v}}{\delta t}, \qquad -\rho\mathbf{F} = \rho\frac{\delta}{\delta t}\big(\epsilon\,\mathbf{v} + B\boldsymbol{\omega}\big). \tag{19}$$

By obtaining Eqs. (19) we used expressions for the momentum densities (10), (11) and the mass balance equation (16). Summing up both sides of Eqs. (19) we obtain the momentum balance equations for the gyrostats

$$\nabla\cdot\boldsymbol{\tau} + \rho\mathbf{f} = \rho\frac{\delta}{\delta t}\big(\mathbf{v} + B\boldsymbol{\omega}\big). \tag{20}$$

2.5 Angular momentum balance equations

Now we formulate the equation of balance of angular momentum (4) for the carrier bodies in control volume V:

$$\frac{d}{dt}\int_{(V)} \rho\mathbf{K}_2^{(cb)}dV = \int_{(V)} \rho\big(\mathbf{r}\times\mathbf{f} + \mathbf{r}\times\mathbf{F} + \mathbf{m}\big)dV + \int_{(S)} \big(\mathbf{r}\times\boldsymbol{\tau}_n + \boldsymbol{\mu}_n\big)dS - \\ - \int_{(S)} (\mathbf{n}\cdot\mathbf{v})\rho\mathbf{K}_2^{(cb)}dS. \tag{21}$$

Here $\mathbf{m}$ is the mass density of external moments acting on the carrier bodies of gyrostats; $\boldsymbol{\mu}_n$ is the moment vector modelling the influence of surrounding medium on the carrier bodies of gyrostats being on surface S of control volume V. Next we formulate the equation of balance of angular momentum (4) for the rotors in control volume V:

$$\frac{d}{dt}\int_{(V)}\rho\mathbf{K}_2^{(rot)}dV=\int_{(V)}\rho\big(-\mathbf{r}\times\mathbf{F}+\mathbf{L}\big)dV+\int_{(S)}\mathbf{T}_n dS-\int_{(S)}(\mathbf{n}\cdot\mathbf{v})\rho\mathbf{K}_2^{(rot)}dS. \quad (22)$$

Here $\mathbf{L}$ is the mass density of external moments acting on the rotors; $\mathbf{T}_n$ is the moment vector modelling the influence of surrounding medium on the rotors of gyrostats on surface S of control volume V.

By standard reasoning we introduce the concept of moment stress tensors $\boldsymbol{\mu}$ and $\mathbf{T}$ which are concerned with the moment vectors $\boldsymbol{\mu}_n$ and $\mathbf{T}_n$ by relations: $\boldsymbol{\mu}_n=\mathbf{n}\cdot\boldsymbol{\mu}$, $\mathbf{T}_n=\mathbf{n}\cdot\mathbf{T}$. Also by standard reasoning we derive the equations of balance of angular momentum in the local form from Eqs. (21), (22). After simple transformations these equations can be written in the form

$$\nabla\cdot\boldsymbol{\mu}+\boldsymbol{\tau}_\times+\rho\mathbf{m}=\rho\frac{\delta}{\delta t}\big(\mathbf{I}_*\cdot\tilde{\boldsymbol{\omega}}\big). \quad (23)$$

$$\nabla\cdot\mathbf{T}+\rho\mathbf{L}=\rho\Big[\mathbf{v}\times B\boldsymbol{\omega}+\frac{\delta}{\delta t}\big(B\mathbf{v}+J\boldsymbol{\omega}\big)\Big]. \quad (24)$$

By obtaining Eqs. (23), (24) we used expression for the angular momentum densities (10), (11), the equation of mass balance (16), and the equations of momentum balance (19). Eq. (23) describes the rotational motion of the carrier bodies and Eq. (24) characterizes the motion of the rotors. It is easy to see that Eqs. (23), (24) do not contain parameter ϵ.

2.6 Equation of energy balance

Now we formulate the equation of energy balance (5) for the material medium in control volume V:

$$\frac{d}{dt}\int_{(V)}\rho(K+U)dV=\int_{(V)}\rho\big(\mathbf{f}\cdot\mathbf{v}+\mathbf{m}\cdot\tilde{\boldsymbol{\omega}}+\mathbf{L}\cdot\boldsymbol{\omega}+Q\big)dV+$$
$$+\int_{(S)}\big(\boldsymbol{\tau}_n\cdot\mathbf{v}+\boldsymbol{\mu}_n\cdot\tilde{\boldsymbol{\omega}}+\mathbf{T}_n\cdot\boldsymbol{\omega}+H_n\big)dS-\int_{(S)}(\mathbf{n}\cdot\mathbf{v})\,\rho(K+U)dS. \quad (25)$$

Here U is the internal energy density per unit mass; Q and H_n are the rate of the energy supply in volume and through surface S respectively. The rate of the energy supply through the surface can be expressed in term of energy-flux vector $\mathbf{H}$ by the formula $H_n=\mathbf{n}\cdot\mathbf{H}$.

By standard reasoning, taking into account the equation of mass balance (16) we transform the equation of energy balance (25) to the local form

$$\rho\frac{\delta}{\delta t}(K+U)=\rho\mathbf{f}\cdot\mathbf{v}+\rho\mathbf{m}\cdot\tilde{\boldsymbol{\omega}}+\rho\mathbf{L}\cdot\boldsymbol{\omega}+(\nabla\cdot\boldsymbol{\tau})\cdot\mathbf{v}+(\nabla\cdot\boldsymbol{\mu})\cdot\tilde{\boldsymbol{\omega}}+$$
$$+(\nabla\cdot\mathbf{T})\cdot\boldsymbol{\omega}+\boldsymbol{\tau}^T\cdot\cdot\nabla\mathbf{v}+\boldsymbol{\mu}^T\cdot\cdot\nabla\tilde{\boldsymbol{\omega}}+\mathbf{T}^T\cdot\cdot\nabla\boldsymbol{\omega}+\nabla\cdot\mathbf{H}+\rho Q. \quad (26)$$

Using expression (14) for the kinetic energy density and the balance equations (20), (23), (24) we transform the energy balance equation (26) to the form

$$\rho \frac{\delta U}{\delta t} = \boldsymbol{\tau}^T \cdot\cdot \big(\nabla \mathbf{v} + \mathbf{E} \times \tilde{\boldsymbol{\omega}}\big) + \boldsymbol{\mu}^T \cdot\cdot \nabla \tilde{\boldsymbol{\omega}} + \mathbf{T}^T \cdot\cdot \nabla \boldsymbol{\omega} + \nabla \cdot \mathbf{H} + \rho Q. \qquad (27)$$

If the supply of energy of "non-mechanical nature" is ignored, i. e. the body under consideration is assumed to be isolated, then Eq. (27) takes a more simple form:

$$\rho \frac{\delta U}{\delta t} = \boldsymbol{\tau}^T \cdot\cdot \big(\nabla \mathbf{v} + \mathbf{E} \times \tilde{\boldsymbol{\omega}}\big) + \boldsymbol{\mu}^T \cdot\cdot \nabla \tilde{\boldsymbol{\omega}} + \mathbf{T}^T \cdot\cdot \nabla \boldsymbol{\omega}. \qquad (28)$$

Below we consider only isolated bodies. Furthermore we are going to derive equations of dynamics, expressions for strain tensors, and constitutive equations in the framework of the geometrically linear theory.

2.7 Linear theory of the elastic medium

Let us consider tensor $\tilde{\mathbf{P}}(\mathbf{r},t)$ (rotation tensor of carrier bodies) and tensor $\mathbf{P}(\mathbf{r},t)$ (rotation tensor of rotors). We assume that in the reference configurations the tensors $\tilde{\mathbf{P}}(\mathbf{r},t)$ and $\mathbf{P}(\mathbf{r},t)$ are equal to the unit tensor. Therefore, upon the linearization near the reference position they take the form

$$\tilde{\mathbf{P}}(\mathbf{r},t) = \mathbf{E} + \boldsymbol{\varphi}(\mathbf{r},t) \times \mathbf{E}, \qquad \mathbf{P}(\mathbf{r},t) = \mathbf{E} + \boldsymbol{\theta}(\mathbf{r},t) \times \mathbf{E}, \qquad (29)$$

where $\boldsymbol{\varphi}(\mathbf{r},t)$, $\boldsymbol{\theta}(\mathbf{r},t)$ are the rotation vector fields of carrier bodies and rotors, respectively. Kinematic relations (8) in the linear approximation are

$$\mathbf{v} = \frac{d\mathbf{u}}{dt}, \qquad \tilde{\boldsymbol{\omega}} = \frac{d\boldsymbol{\varphi}}{dt}, \qquad \boldsymbol{\omega} = \frac{d\boldsymbol{\theta}}{dt}. \qquad (30)$$

The mass balance equation (16) in the linear approximation takes the form

$$\frac{d\rho}{dt} + \rho_* \nabla \cdot \mathbf{v} = 0 \qquad \Rightarrow \qquad \rho = \rho_* \big(1 - \nabla \cdot \mathbf{u}\big). \qquad (31)$$

Here ρ_* is mass density per unit volume in the reference position. Note that mass density at the initial time instant ρ_0 may not coincide with the mass density in the reference position ρ_*. These two quantities are related with each other by the formula

$$\rho_0 = \rho_* \big(1 - \nabla \cdot \mathbf{u}_0\big), \qquad (32)$$

and they coincide only if the material medium is not deformable at the initial time instant.

In view of the above simplifications the equation of motion of the material continuum (20), (23) can be rewritten in the form

$$\nabla \cdot \boldsymbol{\tau} + \rho_* \mathbf{f} = \rho_* \frac{d}{dt}\big(\mathbf{v} + B\boldsymbol{\omega}\big), \qquad \nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \rho_* \mathbf{m} = \rho_* \frac{d}{dt}\big(\mathbf{I}_*^{(0)} \cdot \tilde{\boldsymbol{\omega}}\big), \qquad (33)$$

where inertia tensor $\mathbf{I}_*^{(0)}$ is calculated in the reference configuration. The equation of motion of the rotors (24) takes the form

$$\nabla \cdot \mathbf{T} + \rho_* \mathbf{L} = \rho_* \frac{d}{dt}\big(B\mathbf{v} + J\boldsymbol{\omega}\big), \qquad (34)$$

and after simple transformations the equation of energy balance (28) is written as follows:

$$\rho_* \frac{dU}{dt} = \boldsymbol{\tau}^T \cdot\cdot \frac{d\boldsymbol{\varepsilon}}{dt} + \boldsymbol{\mu}^T \cdot\cdot \frac{d\boldsymbol{\kappa}}{dt} + \mathbf{T}^T \cdot\cdot \frac{d\boldsymbol{\vartheta}}{dt}, \tag{35}$$

where the strain tensors $\boldsymbol{\varepsilon}$, $\boldsymbol{\kappa}$, $\boldsymbol{\vartheta}$ are introduced into consideration. These tensors are calculated by the formulas

$$\boldsymbol{\varepsilon} = \nabla \mathbf{u} + \mathbf{E} \times \boldsymbol{\varphi}, \qquad \boldsymbol{\kappa} = \nabla \boldsymbol{\varphi}, \qquad \boldsymbol{\vartheta} = \nabla \boldsymbol{\theta}. \tag{36}$$

In what follows we consider the elastic material i. e. a material whose density of internal energy and the tensors of force and moment stresses depend only on the strain tensors and do not depend on velocities. For the elastic material the Cauchy–Green relations follow from the equation of energy balance (35):

$$\boldsymbol{\tau} = \rho_* \frac{\partial U}{\partial \boldsymbol{\varepsilon}}, \qquad \boldsymbol{\mu} = \rho_* \frac{\partial U}{\partial \boldsymbol{\kappa}}, \qquad \mathbf{T} = \rho_* \frac{\partial U}{\partial \boldsymbol{\vartheta}}. \tag{37}$$

To close the system of differential equations it is necessary to express the internal energy as a function of the strain tensors

$$\rho_* U = \rho_* U(\boldsymbol{\varepsilon}, \boldsymbol{\kappa}, \boldsymbol{\vartheta}). \tag{38}$$

Now we consider the physically linear theory and therefore we represent the density of internal energy in the following form:

$$\begin{aligned} \rho_* U = \boldsymbol{\tau}_0^T \cdot\cdot \boldsymbol{\varepsilon} + \boldsymbol{\mu}_0^T \cdot\cdot \boldsymbol{\kappa} + \mathbf{T}_*^T \cdot\cdot \boldsymbol{\vartheta} + \frac{1}{2} \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_1 \cdot\cdot \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_2 \cdot\cdot \boldsymbol{\kappa} + \\ + \frac{1}{2} \boldsymbol{\kappa} \cdot\cdot {}^4\mathbf{C}_3 \cdot\cdot \boldsymbol{\kappa} + \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_4 \cdot\cdot \boldsymbol{\vartheta} + \boldsymbol{\kappa} \cdot\cdot {}^4\mathbf{C}_5 \cdot\cdot \boldsymbol{\vartheta} + \frac{1}{2} \boldsymbol{\vartheta} \cdot\cdot {}^4\mathbf{C}_6 \cdot\cdot \boldsymbol{\vartheta}. \end{aligned} \tag{39}$$

Coefficients $\boldsymbol{\tau}_0$, $\boldsymbol{\mu}_0$ and $\mathbf{T}_*$ are called the initial stresses. Coefficients of the quadratic form (39) are called the stiffness tensors. In the linear theory the stiffness tensors do not depend on time. The only restriction imposed on the stiffness tensors is concerned with the requirement of positive definiteness of the quadratic form (39). The structure of the stiffness tensors and the values of the coefficients of elasticity are determined by the physical properties of the material medium.

After substituting expression for the density of internal energy (39) in the Cauchy–Green relations (37) we obtain the following constitutive equations:

$$\begin{aligned} \boldsymbol{\tau}^T &= \boldsymbol{\tau}_0^T + {}^4\mathbf{C}_1 \cdot\cdot \boldsymbol{\varepsilon} + {}^4\mathbf{C}_2 \cdot\cdot \boldsymbol{\kappa} + {}^4\mathbf{C}_4 \cdot\cdot \boldsymbol{\vartheta}, \\ \boldsymbol{\mu}^T &= \boldsymbol{\mu}_0^T + \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_2 + {}^4\mathbf{C}_3 \cdot\cdot \boldsymbol{\kappa} + {}^4\mathbf{C}_5 \cdot\cdot \boldsymbol{\vartheta}, \\ \mathbf{T}^T &= \mathbf{T}_*^T + \boldsymbol{\varepsilon} \cdot\cdot {}^4\mathbf{C}_4 + \boldsymbol{\kappa} \cdot\cdot {}^4\mathbf{C}_5 + {}^4\mathbf{C}_6 \cdot\cdot \boldsymbol{\vartheta}. \end{aligned} \tag{40}$$

According to Eqs. (40) all stress tensors can depend on all strain tensors. It means, in particular, that the moment stress tensor of rotors can depend not only on their relative orientation, but also on the relative orientation and relative position of the carrier bodies.

3 Continuum consisting of multi-rotor gyrostats

3.1 Kinematics of the continuum

The multi-rotor gyrostat (see Fig. 4) is a complex object which consists of the carrier body and N rotors. The carrier body and the rotors of the gyrostat are considered to be the infinitesimal rigid bodies. The carrier body has an arbitrary tensor of inertia. The rotors are the axisymmetric rigid bodies. The axes of symmetry of the rotors are fixed with respect to the carrier body. The rotors can not translate relative to the carrier body, and they can rotate independently of rotation of the carrier body only about their axes of symmetry.

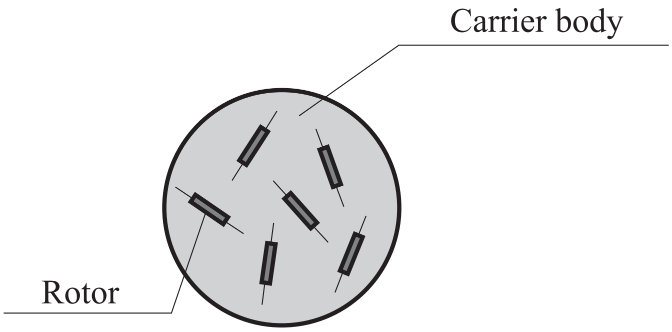

Figure 4: Multi-rotor gyrostat

Now we consider the material medium (see Fig. 5) consisting of multi-rotor gyrostats. Let vector $\mathbf{r}$ determine the position of some point of space. We introduce following notations: $\mathbf{v}(\mathbf{r}, t)$ is the velocity field; $\mathbf{u}(\mathbf{r}, t)$ is the displacement field; $\mathbf{P}_0(\mathbf{r}, t)$, $\boldsymbol{\omega}_0(\mathbf{r}, t)$ are the fields of the rotation tensors and the angular velocity vectors of the carrier bodies; $\mathbf{P}_i(\mathbf{r}, t)$, $\boldsymbol{\omega}_i(\mathbf{r}, t)$ are fields of the rotation tensor and the angular velocity vector of the rotor number i, where $i = 1, 2, ..., N$.

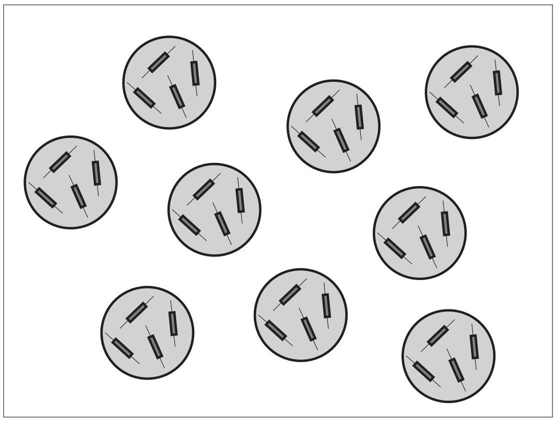

Figure 5: Elementary volume of continuum consisting of multi-rotor gyrostats

In view of the fact that the rotor number i can rotate independently of rotation of the carrier body only about its axis of symmetry the rotation tensor of the rotor is represented in the form

$$\mathbf{P}_i(\mathbf{r}, t) = \mathbf{P}_0(\mathbf{r}, t) \cdot \mathbf{P}(\beta_i \mathbf{n}_i), \qquad (41)$$

where $\mathbf{n}_i$ is the unit vector which determines the direction of the axis of symmetry of the rotor at the reference position, $\beta_i(\mathbf{r}, t)$ is the angle of rotation of the rotor with respect to the carrier body. Hence the unit vector $\mathbf{n}'_i(\mathbf{r}, t)$ which determines the direction of the axis of symmetry of the rotor number i at the actual position takes the form

$$\mathbf{n}'_i(\mathbf{r}, t) = \mathbf{P}_0(\mathbf{r}, t) \cdot \mathbf{n}_i. \tag{42}$$

In the spatial description the angular velocity vector of the carrier body and angular velocity vectors of the rotors are calculated by the formulas:

$$\boldsymbol{\omega}_0 = -\frac{1}{2}\left(\frac{\delta \mathbf{P}_0}{\delta t} \cdot \mathbf{P}_0^T\right)_\times, \qquad \boldsymbol{\omega}_i = \boldsymbol{\omega}_0 + \frac{\delta \beta_i}{\delta t}\,\mathbf{n}'_i, \qquad i = 1, 2, ..., N. \tag{43}$$

3.2 The equations of motion

The multi-rotor gyrostat has $N + 6$ degrees of freedom which are determined by the following functions:

$$\mathbf{v}(\mathbf{r}, t), \quad \mathbf{P}_0(\mathbf{r}, t), \quad \beta_i(\mathbf{r}, t), \quad i = 1, 2, ..., N. \tag{44}$$

In order to find these unknown functions we need to formulate two vector and N scalar equations of motion.

The momentum balance equation for the gyrostats has the form

$$\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{f} = \rho \frac{\delta \mathbf{v}}{\delta t}. \tag{45}$$

Here $\boldsymbol{\tau}$ is the stress tensor, $\mathbf{f}$ is the mass density of external forces, ρ is the mass density which satisfies Eq. (16).

The angular momentum balance equation for the gyrostats is

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_\times + \rho \mathbf{m} = \rho \frac{\delta}{\delta t}\,\boldsymbol{\mathfrak{L}}(\mathbf{r}, t), \tag{46}$$

where $\boldsymbol{\mu}$ is the moment stress tensor, $\mathbf{m}$ is the mass density of external moments, $\boldsymbol{\mathfrak{L}}$ is the mass density of the proper angular momentum of the gyrostat

$$\boldsymbol{\mathfrak{L}} = \mathbf{P}_0(\mathbf{r},t) \cdot \mathbf{C} \cdot \mathbf{P}_0^T(\mathbf{r},t) \cdot \boldsymbol{\omega}_0(\mathbf{r},t) + \sum_{i=1}^{N} \lambda_i \frac{\delta \beta_i(\mathbf{r},t)}{\delta t}\,\mathbf{n}'_i(\mathbf{r},t). \tag{47}$$

The first term on the right-hand side of Eq. (47) is the mass density of the proper angular momentum of the gyrostats when the gyrostats move as the rigid bodies, i.e. all rotors are fixed with respect to the carrier body of the gyrostat. The rest terms on the right-hand side of Eq. (47) characterize the influence of the independent rotation of rotors on the proper angular momentum of the gyrostat. Tensor $\mathbf{C}$ is the inertia tensor of the gyrostat at the reference configuration, λ_i is the axial moment of inertia if the rotor number i.

We should add the system of equations (45), (46) to the the angular momentum balance equations for the rotors. The projections of these equations on the axes of the rotors are

$$\lambda_i \frac{\delta}{\delta t}\left(\frac{\delta \beta_i(\mathbf{r},t)}{\delta t} + \boldsymbol{\omega}_0(\mathbf{r},t) \cdot \mathbf{n}'_i(\mathbf{r},t)\right) = L_i, \qquad i = 1, 2, ..., N. \tag{48}$$

Here L_i is the mass density of external moments acting on the rotor number i. The moment interaction between rotors is supposed to be zero. That is why the moment stress tensors are absent in Eqs. (48).

Now we have two vector and N scalar equations of motion. The system of equations is not closed. It is necessary to add these equations to the constitutive equations. In order to obtain the constitutive equations we should consider the equation of energy balance.

3.3 Equation of energy balance

Now we formulate the equation of energy balance (5) for the material medium in control volume V:

$$\frac{d}{dt}\int\limits_{(V)} \rho(K+U)dV = \int\limits_{(V)} \rho\big(\mathbf{f}\cdot\mathbf{v} + \mathbf{m}\cdot\boldsymbol{\omega}_0 + Q\big)dV + \\ + \int\limits_{(S)} \big(\boldsymbol{\tau}_n\cdot\mathbf{v} + \boldsymbol{\mu}_n\cdot\boldsymbol{\omega}_0 + H_{(n)}\big)\, dS - \int\limits_{(S)} \rho\mathbf{n}\cdot\mathbf{v}(K+U)dS. \quad (49)$$

Here U is the internal energy density per unit mass; $\boldsymbol{\tau}_n = \mathbf{n}\cdot\boldsymbol{\tau}$, $\boldsymbol{\mu}_n = \mathbf{n}\cdot\boldsymbol{\mu}$; quantities Q and H_n are the rate of the energy supply in volume and through surface S respectively. The rate of the energy supply through the surface can be expressed in term of energy-flux vector $\mathbf{H}$ by the formula $H_n = \mathbf{n}\cdot\mathbf{H}$. The mass density of kinetic energy for gyrostat has the form

$$K = \frac{1}{2}\mathbf{v}\cdot\mathbf{v} + \frac{1}{2}\boldsymbol{\omega}_0\cdot\mathbf{P}_0\cdot\mathbf{C}\cdot\mathbf{P}_0^T\cdot\boldsymbol{\omega}_0 + \frac{1}{2}\sum_{i=1}^{N}\lambda_i\left[\left(\frac{\delta\beta_i}{\delta t}\right)^2 + 2\,\frac{\delta\beta_i}{\delta t}\,\boldsymbol{\omega}_0\cdot\mathbf{n}_i'\right]. \quad (50)$$

Remark. *From the angular momentum balance equation (46) it is not clear the sense of the external moment* $\mathbf{m}$ *and the moment stress tensor* $\boldsymbol{\mu}$*. These moments are the summary moments acting on the gyrostats. In fact, they can act on the carrier bodies or on the rotors. It is not important for Eq. (46). But it is very important to take into account the sense of* $\mathbf{m}$ *and* $\boldsymbol{\mu}$ *when we formulate the energy balance equation. The reason is the fact that calculating the power of external actions we should multiply the moments acting on the carrier bodies by* $\boldsymbol{\omega}_0$ *and the moments acting on the rotors by* $\boldsymbol{\omega}_i$*. From Eq. (49) it is obvious that* $\mathbf{m}$ *and* $\boldsymbol{\mu}$ *are supposed to act on the carrier bodies. Since the moments* L_i *are absent in Eq. (49) it is clear that they are the internal moments characterizing the interaction of the rotors and the carrier bodies.*

By standard reasoning, taking into account the equation of mass balance (16) we transform the equation of energy balance (49) to the local form

$$\rho\frac{\delta}{\delta t}(K+U) = \rho\mathbf{f}\cdot\mathbf{v} + \rho\mathbf{m}\cdot\boldsymbol{\omega}_0 + (\nabla\cdot\boldsymbol{\tau})\cdot\mathbf{v} + (\nabla\cdot\boldsymbol{\mu})\cdot\boldsymbol{\omega}_0 + \\ + \boldsymbol{\tau}^T\cdot\cdot\nabla\mathbf{v} + \boldsymbol{\mu}^T\cdot\cdot\nabla\boldsymbol{\omega}_0 + \nabla\cdot\mathbf{H} + \rho Q. \quad (51)$$

Using expression (50) for the kinetic energy density and Eqs. (45)–(48) we transform the energy balance equation (51) to the form

$$\rho \frac{\delta U}{\delta t} = \boldsymbol{\tau}^T \cdot\cdot (\nabla \mathbf{v} + \mathbf{E} \times \boldsymbol{\omega}_0) + \boldsymbol{\mu}^T \cdot\cdot \nabla \boldsymbol{\omega}_0 + \nabla \cdot \mathbf{H} + \rho Q - \rho \sum_{i=1}^{N} L_i \frac{\delta \beta_i}{\delta t}. \qquad (52)$$

Notice that the last term on the right-hand side of of Eq. (52) plays role analogous to the term ρQ. It has the sense of the rate of energy supply. In the case of the elastic continuum the subsequent procedure of derivation of the Cauchy–Green relations is standard for continuum mechanics.

4 Concluding remarks

The method of modelling of continuum is as follows:

- We should choose the main variables, write down kinematics relations and determine the dynamical structure of the continuum.
- We should write down the law of mass conservation.
- We should formulate the momentum balance equation and the angular momentum balance equation.
- We should formulate the energy balance equation and obtain the constitutive equations.

The main features of the method of modelling of continuum in the case when the continuum possesses the inner rotational degrees of freedom are:

- It is necessary to formulate the additional balance equations. There are two variants. First: we formulate the angular momentum balance equations for all rotors and the carrier body. Second: we formulate the angular momentum balance equations for all rotors and the gyrostat as a whole.
- The energy balance equation contains the additional terms. Some of them can be considered as the rate of energy supply.
- The presence of additional degrees of freedom allows us to use these continuum models for description some "non-mechanical" processes.