Ferromagnets and Kelvin's Medium: Basic Equations and Magnetoacoustic Resonance*

Abstract

The nonlinear constitutive equations of Kelvin's medium (polar medium consisting of rotating particles) are obtained. It is shown that they include the known constitutive equations of ferromagnetic insulators as a particular case. Another way of taking the couplings of magnetic and elastic subsystems into account is suggested. Wave processes are investigated from this point of view. All results are interpreted both in terms of mechanical medium and ferromagnets.

1 Introduction

There are a lot of papers devoted to theories of elastic polar media. The first investigation in this field was developed by E. Cosserat and F. Cosserat [1]. Each particle of such a medium is a small rigid body (a point body). In [2], [3], [4] the linear theory for infinitesimal turns and displacements is considered.

In this paper we obtain a general form for *nonlinear* constitutive equations for Cosserat medium. Then we consider a special case of this medium — Kelvin's medium. Kelvin's medium is an elastic polar medium consisting of rotating particles with axial symmetry (Fig. 1). These particles can oscillate and rotate in general ways. Point bodies of this medium contrary to Cosserat continuum may have large angular velocities; displacements and turns may be finite. The idea to consider such a continuum was suggested by Lord Kelvin: "Kelvin imagined a model of a quasi-rigid ether built from gyrostates. The problem was to find a system resisting only to deformations concerned with rotation" [5].

We suppose that internal forces in this continuum do not depend on the angles of own rotation of particles or angular velocities of own rotation. We obtain the constitutive equations of this medium using the law of balance of energy via a phenomenological method suggested in [6]. This method allows to get nonlinear constitutive equations for elastic polar medium with particles of general kind, i.e. for generalized Cosserat medium.

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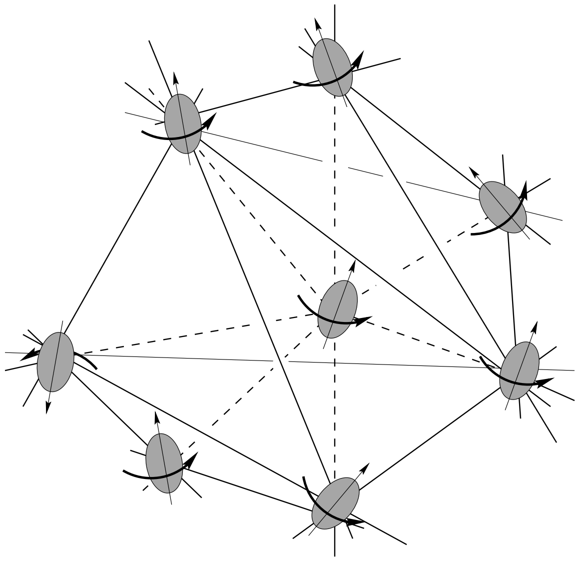

Figure 1: Kelvin's medium

We obtain these equations in section 2. Afterwards we take into account restrictions given by axial symmetry of particles and get constitutive equations of Kelvin's medium. We found these equations to be analogous to the constitutive equations of saturated elastic ferromagnetic insulators (see [7]) and to the constitutive equations in the non-classical theory of elastic shells [6]. There is also an exact analogy between dynamic equations of ferromagnets [7] and Kelvin's medium. This carries a similarity of wave processes in both media. We use the most general way of taking into account the coupling of translational and angular deformations in the function of strain energy. This allows us to describe phenomena analogous to magnetoacoustic resonance in ferromagnetic materials.

2 Dynamic and constitutive equations of Kelvin's medium

2.1 Kinematics of Kelvin's medium

We shall consider a deformable medium consisting of rotating particles with rotational symmetry having both translational and angular degrees of freedom.

Let q^s be material coordinates of a point of this medium, $\mathbf{r}(q^s)$ and $\mathbf{R}(q^s)$ are radius vectors of centre of mass of a point body in the initial and actual configuration respectively. [Here and further Roman subscripts take values 1,2,3 and Greek ones 1,2 and we shall employ the usual summation convention.] Let us associate with each point of this continuum an orthonormal vector basis $\mathbf{D}_k(q^s)$ that is "frozen" into a point body, where $\mathbf{m} \equiv \mathbf{D}_3$ is a unit vector of an axis of a point body. In the initial configuration let $\mathbf{D}_k = \mathbf{d}_k$, $\mathbf{m}_0 \equiv \mathbf{d}_3$. The dual basis $\mathbf{D}^k$ ($\mathbf{D}^k \cdot \mathbf{D}_i = \delta_i^k$) coincides with $\mathbf{D}_k$. We may introduce a turn-tensor $\mathbf{P} = \mathbf{D}_k \otimes \mathbf{d}^k$ that describes the turn of a point body. One can see that $\mathbf{D}_k = \mathbf{P} \cdot \mathbf{d}_k$. It is easy to show that $\mathbf{P} \cdot \mathbf{P}^\top = \mathbf{E}$, where $\mathbf{E}$ is a unit tensor, and that $\det \mathbf{P} = 1$. The turn-tensor can be represented in the form:

$$\mathbf{P}(t) = \mathbf{P}_3(\psi \mathbf{m}_0) \cdot \mathbf{P}_2(\vartheta \mathbf{l}_0) \cdot \mathbf{P}_1(\varphi \mathbf{m}_0), \tag{1}$$

where $\mathbf{P}_3(\psi \mathbf{m}_0) = (1 - \cos\psi)\mathbf{m}_0 \otimes \mathbf{m}_0 + \cos\psi \mathbf{E} + \sin\psi \mathbf{m}_0 \times \mathbf{E}$ is a turn-tensor about an axis $\mathbf{m}_0$ about an angle ψ etc., $\mathbf{l}_0$ and $\mathbf{m}_0$ are orthonormal vectors, ψ, ϑ, φ are angles of precession, nutation and own rotation respectively. We see that
$\mathbf{m}_0 \cdot \mathbf{P}_3 = \mathbf{m}_0 = \mathbf{P}_3 \cdot \mathbf{m}_0$ etc.

Let us denote $\mathbf{r}_i = \dfrac{\partial \mathbf{r}}{\partial q^i} \equiv \partial_i \mathbf{r}$ and $\mathbf{R}_i = \dfrac{\partial \mathbf{R}}{\partial q^i} \equiv \partial_i \mathbf{R}$. Nabla operators in the initial and actual configuration are defined by $\overset{\circ}{\nabla} = \mathbf{r}^i \partial_i$ and $\nabla = \mathbf{R}^i \partial_i$ respectively, where $\mathbf{r}^i$ and $\mathbf{R}^i$ are corresponding dual bases. We suppose $\overset{\circ}{\nabla} \mathbf{m}_0 = \mathbf{0}$ and put $\mathbf{r}^3 = \mathbf{m}_0$.

Let us introduce the following notation:

$\mathbf{u} = \mathbf{R} - \mathbf{r}$ is the translational displacement of a centre of mass of a point body;

$\mathbf{v} = \dot{\mathbf{R}}$ is the velocity of a centre of mass of a point body;

$\boldsymbol{\omega}(\mathbf{R}, t)$ is the angular velocity of a point body;

it can be defined by Poisson equation

$$\dot{\mathbf{P}} = \boldsymbol{\omega} \times \mathbf{P}; \tag{2}$$

and can be calculated as

$$\boldsymbol{\omega} = -[\dot{\mathbf{P}} \cdot \mathbf{P}^\top]_\times / 2; \tag{3}$$

or as

$$\boldsymbol{\omega} = \dot{\psi} \mathbf{m}_0 + \dot{\vartheta} \mathbf{P}_3 \cdot \mathbf{l}_0 + \dot{\varphi} \mathbf{P}_3 \cdot \mathbf{P}_2 \cdot \mathbf{m}_0 = \dot{\psi} \mathbf{m}_0 + \dot{\vartheta} \mathbf{l} + \dot{\varphi} \mathbf{m}, \tag{4}$$

where $\mathbf{l} = \mathbf{P}_3 \cdot \mathbf{l}_0$. One can see that $\dot{\mathbf{P}}_3 = \dot{\psi} \mathbf{m}_0 \times \mathbf{P}_3$ etc., and that
$\mathbf{l} \cdot \mathbf{m} = \mathbf{l}_0 \cdot \mathbf{P}_3^\top \cdot \mathbf{P}_3 \cdot \mathbf{P}_2 \cdot \mathbf{m}_0 = \mathbf{l}_0 \cdot \mathbf{m}_0 = 0$.

Let ut write an analog of Poisson equation for coordinate q^i instead of time t:

$$\partial_i \mathbf{P} = \boldsymbol{\Phi}_i \times \mathbf{P}, \tag{5}$$

here $\boldsymbol{\Phi}_i$ can be found as

$$\boldsymbol{\Phi}_i = -[\partial_i \mathbf{P} \cdot \mathbf{P}^\top]_\times / 2; \tag{6}$$

or as

$$\boldsymbol{\Phi}_i = \partial_i \psi \mathbf{m}_0 + \partial_i \vartheta \mathbf{P}_3 \cdot \mathbf{l}_0 + \partial_i \varphi \mathbf{P}_3 \cdot \mathbf{P}_2 \cdot \mathbf{m}_0 = \partial_i \psi \mathbf{m}_0 + \partial_i \vartheta \mathbf{l} + \partial_i \varphi \mathbf{m}. \tag{7}$$

Further we shall use relation

$$\partial_i \boldsymbol{\omega} = \dot{\boldsymbol{\Phi}}_i + \boldsymbol{\Phi}_i \times \boldsymbol{\omega} \tag{8}$$

Proof:

$$\begin{aligned}\partial_i\boldsymbol{\omega} = -\partial_i[\dot{\mathbf{P}}\cdot\mathbf{P}^\top]_\times/2 = -[(\partial_i\dot{\mathbf{P}})\cdot\mathbf{P}^\top + \dot{\mathbf{P}}\cdot\partial_i\mathbf{P}^\top]_\times/2 =\\ = -[(\boldsymbol{\Phi}_i\times\mathbf{P})^{\cdot}\cdot\mathbf{P}^\top + \boldsymbol{\omega}\times\mathbf{P}\cdot(\boldsymbol{\Phi}_i\times\mathbf{P})^\top]_\times/2 =\\ = -[\dot{\boldsymbol{\Phi}}_i\times\mathbf{E} + \boldsymbol{\Phi}_i\times(\boldsymbol{\omega}\times\mathbf{P})\cdot\mathbf{P}^\top - \boldsymbol{\omega}\times\mathbf{P}\cdot\mathbf{P}^\top\times\boldsymbol{\Phi}_i]_\times/2 =\\ = -[\dot{\boldsymbol{\Phi}}_i\times\mathbf{E} + \boldsymbol{\Phi}_i\times(\boldsymbol{\omega}\times\mathbf{E}) - \boldsymbol{\omega}\times\mathbf{E}\times\boldsymbol{\Phi}_i]_\times/2 = \dot{\boldsymbol{\Phi}}_i + \boldsymbol{\Phi}_i\times\boldsymbol{\omega}.\end{aligned} \tag{9}$$

Here we used (3) and (6).

Let us introduce strain tensors

$$\begin{aligned}\mathbf{A} &= \overset{\circ}{\nabla}\mathbf{R}\cdot\mathbf{P},\\ \mathbf{K} &= \mathbf{r}^i\otimes\boldsymbol{\Phi}_i\cdot\mathbf{P}.\end{aligned} \tag{10}$$

Tensor $\mathbf{A}$ is responsible both for translational and angular deformation, and $\mathbf{K}$ is determined only by angular strain; using (7) we may get

$$\mathbf{K} = \overset{\circ}{\nabla}\psi\mathbf{m}_0\cdot\mathbf{P} + \overset{\circ}{\nabla}\vartheta\mathbf{l}\cdot\mathbf{P} + \overset{\circ}{\nabla}\varphi\mathbf{m}_0; \tag{11}$$

one can show that $\mathbf{K} = -(\overset{\circ}{\nabla}\mathbf{P}\cdot\mathbf{P}^\top)\cdot\cdot(\mathbf{E}\times\mathbf{P})/2$.

The density of kinetic energy is defined by

$$\mathcal{K} = \frac{1}{2}(\mathbf{v}\cdot\mathbf{v} + \boldsymbol{\omega}\cdot\boldsymbol{\Theta}\cdot\boldsymbol{\omega}), \tag{12}$$

where $\boldsymbol{\Theta} = \mathbf{P}\cdot\boldsymbol{\Theta}_0\cdot\mathbf{P}^\top$ is the density of central inertia tensor of a point body in the actual configuration, $\boldsymbol{\Theta}_0$ is the density of central inertia tensor of a point body in the initial configuration; since point bodies have an axial symmetry,

$$\boldsymbol{\Theta} = \lambda\mathbf{m}\otimes\mathbf{m} + \mu(\mathbf{E} - \mathbf{m}\otimes\mathbf{m}). \tag{13}$$

The density of impulse of the medium is given by the formula

$$\mathcal{K}_1 = \frac{\partial\mathcal{K}}{\partial\mathbf{v}} = \mathbf{v}, \tag{14}$$

and the density of kinetic moment calculated relatively to the origin is

$$\mathcal{K}_2 = \frac{\partial\mathcal{K}}{\partial\boldsymbol{\omega}} + \mathbf{R}\times\mathcal{K}_1 = \boldsymbol{\Theta}\cdot\boldsymbol{\omega} + \mathbf{R}\times\mathbf{v}. \tag{15}$$

2.2 Stress and couple tensors. Euler's laws of dynamics.

We denote

$\rho(\mathbf{R}, t)$ is the mass density in the actual configuration;

$\boldsymbol{\tau}(\mathbf{R}, t)$ is Cauchy stress-tensor; $\boldsymbol{\tau}_{(\mathbf{n})} = \mathbf{n}\cdot\boldsymbol{\tau}$, where $\boldsymbol{\tau}_{(\mathbf{n})}$ is stress vector acting upon the elementary surface, $\mathbf{n}$ is the normal to this surface;

$\boldsymbol{\mu}(\mathbf{R}, t)$ is the couple tensor which can be introduced analogously to the stress-tensor; $\boldsymbol{\mu}_{(\mathbf{n})} = \mathbf{n} \cdot \boldsymbol{\mu}$, where $\boldsymbol{\mu}_{(\mathbf{n})}$ is the moment acting upon the elementary surface with the normal $\mathbf{n}$;

$\mathbf{Q}(\mathbf{R}, t)$ is the density of the external force;

$\mathbf{L}(\mathbf{R}, t)$ is the density of the external moment;

$\mathcal{U}(\mathbf{R}, t)$ is the density of the strain energy.

Euler's first law of dynamics (balance of force) for a part of continuum $\triangle V$ bounded by a surfase Σ is

$$\frac{d}{dt} \int\limits_{\triangle V} \rho\, \mathcal{K}_1 \, dV = \int\limits_{\triangle V} \rho\, \mathbf{Q} \, dV + \int\limits_{\Sigma} \boldsymbol{\tau}_{(\mathbf{n})} \, d\Sigma. \tag{16}$$

It can be rewritten in a local form

$$\nabla \cdot \boldsymbol{\tau} + \rho\, \mathbf{Q} = \rho\, \ddot{\mathbf{u}}. \tag{17}$$

Euler's second law of dynamics (balance of moment) for a part of continuum $\triangle V$ bounded by a surfase Σ is

$$\frac{d}{dt} \int\limits_{\triangle V} \rho\, \mathcal{K}_2 \, dV = \int\limits_{\triangle V} \rho\, (\mathbf{L} + \mathbf{R} \times \mathbf{Q}) \, dV + \int\limits_{\Sigma} (\boldsymbol{\mu}_{(\mathbf{n})} + \mathbf{R} \times \boldsymbol{\tau}_{(\mathbf{n})}) \, d\Sigma. \tag{18}$$

One can rewrite it in a local form using Euler's first law of dynamics (16)

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_{\times} + \rho\, \mathbf{L} = \rho\, (\boldsymbol{\Theta} \cdot \boldsymbol{\omega})^{\cdot}. \tag{19}$$

If we consider the case when the densities of moments of inertia λ and μ are infinitesimal but the angular velocity of own rotation $\dot{\varphi}$ is large so that $\lambda\dot{\varphi} = O(1)$, (19) can be rewritten as

$$\nabla \cdot \boldsymbol{\mu} + \boldsymbol{\tau}_{\times} + \rho\, \mathbf{L} = \rho\, \boldsymbol{\omega} \times (\boldsymbol{\Theta} \cdot \boldsymbol{\omega}) + o\,(1). \tag{20}$$

Proof:

$$\begin{aligned} (\boldsymbol{\Theta} \cdot \boldsymbol{\omega})^{\cdot} = (\mathbf{P} \cdot \boldsymbol{\Theta}_0 \cdot \mathbf{P}^{\top} \cdot \boldsymbol{\omega})^{\cdot} \overset{(2)}{=} (\boldsymbol{\omega} \times \mathbf{P} \cdot \boldsymbol{\Theta}_0 \cdot \mathbf{P}^{\top} - \mathbf{P} \cdot \boldsymbol{\Theta}_0 \cdot \mathbf{P}^{\top} \times \boldsymbol{\omega}) \cdot \boldsymbol{\omega} + \\ + \mathbf{P} \cdot \boldsymbol{\Theta}_0 \cdot \mathbf{P}^{\top} \cdot \dot{\boldsymbol{\omega}} = \boldsymbol{\omega} \times (\boldsymbol{\Theta} \cdot \boldsymbol{\omega}) + \boldsymbol{\Theta} \cdot \dot{\boldsymbol{\omega}} \end{aligned} \tag{21}$$

Since we suppose all interactions to be potential, there can not exist any internal stresses that induce own rotation of a point body, and $\ddot{\varphi}$ is of the same order as $\mathbf{L}$. Taking this in account we may write

$$\begin{aligned} \boldsymbol{\Theta} \cdot \dot{\boldsymbol{\omega}} &\overset{(4)}{=} \boldsymbol{\Theta} \cdot (\ddot{\psi}\mathbf{m}_0 + \ddot{\vartheta}\mathbf{l} + \ddot{\varphi}\mathbf{m} + \dot{\vartheta}\dot{\psi}\mathbf{m}_0 \times \mathbf{l} + \dot{\varphi}\, \boldsymbol{\omega} \times \mathbf{m}) \overset{(13)}{=} \\ &= (\lambda\mathbf{m} \otimes \mathbf{m} + \mu(\mathbf{E} - \mathbf{m} \otimes \mathbf{m})) \cdot (\ddot{\psi}\mathbf{m}_0 + \ddot{\vartheta}\mathbf{l} + \ddot{\varphi}\mathbf{m} + \dot{\vartheta}\dot{\psi}\mathbf{m}_0 \times \mathbf{l} + \dot{\varphi}\, \boldsymbol{\omega} \times \mathbf{m}) = \\ &= \mu\dot{\varphi}\, \boldsymbol{\omega} \times \mathbf{m} + o\,(1) \overset{(4)}{=} \mu\dot{\varphi}(\dot{\psi}\mathbf{m}_0 \times \mathbf{m} + \dot{\vartheta}\mathbf{l} \times \mathbf{m}) + o\,(1) = \\ &= O(1)(\dot{\psi}\mathbf{m}_0 \times \mathbf{m} + \dot{\vartheta}\mathbf{l} \times \mathbf{m}) + o\,(1) = o\,(1) \end{aligned} \tag{22}$$

It is easy to show that under these conditions $\boldsymbol{\omega} \times \boldsymbol{\Theta} \cdot \boldsymbol{\omega} = O(1)$. Indeed,
$\boldsymbol{\omega} \times \boldsymbol{\Theta} \cdot \boldsymbol{\omega} \overset{(13)}{=} \boldsymbol{\omega} \times (\lambda - \mu)\mathbf{m} \otimes \mathbf{m} \cdot \boldsymbol{\omega} + \mu \boldsymbol{\omega} \times \boldsymbol{\omega} \overset{(4)}{=} (\lambda - \mu)(\dot{\varphi} + \dot{\psi}\mathbf{m}_0 \cdot \mathbf{m})(\dot{\psi}\mathbf{m}_0 \times \mathbf{m} + \dot{\vartheta}\mathbf{l} \times \mathbf{m})$

which is $O(1)$ in general case since $\lambda\dot{\varphi} = O(1)$.

Thus we have $(\boldsymbol{\Theta} \cdot \boldsymbol{\omega})^{\cdot} = \boldsymbol{\omega} \times (\boldsymbol{\Theta} \cdot \boldsymbol{\omega}) + o(1)$ and (19) may be rewritten as (20) provided $\lambda = o(1)$, $\mu = o(1)$, $\lambda\dot{\varphi} = O(1)$, $\dot{\psi} = O(1)$, $\dot{\vartheta} = O(1)$.

NB: Under these conditions

$$\boldsymbol{\Theta} \cdot \boldsymbol{\omega} = \lambda\dot{\varphi}\mathbf{m} + \boldsymbol{\Theta} \cdot (\dot{\psi}\mathbf{m}_0 + \dot{\vartheta}\mathbf{l}) = \lambda\dot{\varphi}\mathbf{m} + o(1). \tag{23}$$

2.3 Nonlinear constitutive equations

2.3.1 Nonlinear constitutive equations for generalized Cosserat medium

We obtain the nonlinear constitutive equations for elastic polar medium via the method used in the theory of shells (P.A. Zhilin, [6]).

The equation for balance of energy for a polar medium is:

$$\frac{d}{dt}\int\limits_{\triangle V} \rho(\mathcal{K} + \mathcal{U})\, dV = \int\limits_{\triangle V} \rho(\mathbf{Q} \cdot \mathbf{v} + \mathbf{L} \cdot \boldsymbol{\omega})\, dV + \int\limits_{\Sigma} (\boldsymbol{\tau}_{(\mathbf{n})} \cdot \mathbf{v} + \boldsymbol{\mu}_{(\mathbf{n})} \cdot \boldsymbol{\omega})\, d\Sigma. \tag{24}$$

Its local form is

$$\rho\dot{\mathcal{U}} = \boldsymbol{\tau}^{\top} \cdot\cdot \nabla\mathbf{v} - \boldsymbol{\tau}_{\times} \cdot \boldsymbol{\omega} + \boldsymbol{\mu}^{\top} \cdot\cdot \nabla\boldsymbol{\omega}. \tag{25}$$

It can be rewritten in the form

$$\rho\dot{\mathcal{U}} = \boldsymbol{\tau}_*^{\top} \cdot\cdot \dot{\mathbf{A}} + \boldsymbol{\mu}_*^{\top} \cdot\cdot \dot{\mathbf{K}}, \tag{26}$$

where $\boldsymbol{\tau}_* = \overset{\circ}{\nabla}\mathbf{R}^{-\top} \cdot \boldsymbol{\tau} \cdot \mathbf{P}$ is the energetical stress tensor, $\boldsymbol{\mu}_* = \overset{\circ}{\nabla}\mathbf{R}^{-\top} \cdot \boldsymbol{\mu} \cdot \mathbf{P}$ is the energetical couple tensor.
Proof:

$$\begin{aligned}
&\boldsymbol{\tau}_*^{\top} \cdot\cdot \dot{\mathbf{A}} + \boldsymbol{\mu}_*^{\top} \cdot\cdot \dot{\mathbf{K}} \overset{(10)}{=} (\mathbf{P}^{\top} \cdot \boldsymbol{\tau}^{\top} \cdot (\overset{\circ}{\nabla}\mathbf{R})^{-1}) \cdot\cdot (\overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{P})^{\cdot} + (\mathbf{P}^{\top} \cdot \boldsymbol{\mu}^{\top} \cdot (\overset{\circ}{\nabla}\mathbf{R})^{-1}) \cdot\cdot (\mathbf{r}^i \otimes \boldsymbol{\Phi}_i \cdot \mathbf{P})^{\cdot} \overset{(2)}{=} \\
&= (\mathbf{P}^{\top} \cdot \boldsymbol{\tau}^{\top} \cdot (\overset{\circ}{\nabla}\mathbf{R})^{-1}) \cdot\cdot (\overset{\circ}{\nabla}\mathbf{v} \cdot \mathbf{P}) + (\mathbf{P}^{\top} \cdot \boldsymbol{\tau}^{\top} \cdot (\overset{\circ}{\nabla}\mathbf{R})^{-1}) \cdot\cdot (\overset{\circ}{\nabla}\mathbf{R} \cdot (\boldsymbol{\omega} \times \mathbf{P})) + \\
&+ (\mathbf{P}^{\top} \cdot \boldsymbol{\mu}^{\top} \cdot (\overset{\circ}{\nabla}\mathbf{R})^{-1}) \cdot\cdot (\mathbf{r}^i \otimes \dot{\boldsymbol{\Phi}}_i \cdot \mathbf{P}) + (\mathbf{P}^{\top} \cdot \boldsymbol{\mu}^{\top} \cdot (\overset{\circ}{\nabla}\mathbf{R})^{-1}) \cdot\cdot (\mathbf{r}^i \otimes \boldsymbol{\Phi}_i \cdot (\boldsymbol{\omega} \times \mathbf{P})) = \\
&= \boldsymbol{\tau}^{\top} \cdot\cdot ((\overset{\circ}{\nabla}\mathbf{R})^{-1} \cdot \overset{\circ}{\nabla}\mathbf{v}) + \boldsymbol{\tau}^{\top} \cdot\cdot ((\overset{\circ}{\nabla}\mathbf{R})^{-1} \cdot \overset{\circ}{\nabla}\mathbf{R} \cdot (\boldsymbol{\omega} \times \mathbf{E})) + \boldsymbol{\mu}^{\top} \cdot\cdot ((\overset{\circ}{\nabla}\mathbf{R})^{-1} \cdot \mathbf{r}^i \otimes \dot{\boldsymbol{\Phi}}_i) + \\
&+ \boldsymbol{\mu}^{\top} \cdot\cdot ((\overset{\circ}{\nabla}\mathbf{R})^{-1} \cdot \mathbf{r}^i \otimes \boldsymbol{\Phi}_i \cdot (\boldsymbol{\omega} \times \mathbf{E})) = \boldsymbol{\tau}^{\top} \cdot\cdot \nabla\mathbf{v} + \boldsymbol{\tau}^{\top} \cdot\cdot (\boldsymbol{\omega} \times \mathbf{E}) + \boldsymbol{\mu}^{\top} \cdot\cdot \mathbf{R}^i \otimes (\dot{\boldsymbol{\Phi}}_i + \boldsymbol{\Phi}_i \times \boldsymbol{\omega}) \overset{(8)}{=} \\
&= \boldsymbol{\tau}^{\top} \cdot\cdot \nabla\mathbf{v} - \boldsymbol{\tau}_{\times} \cdot \boldsymbol{\omega} + \boldsymbol{\mu}^{\top} \cdot\cdot \nabla\boldsymbol{\omega}
\end{aligned} \tag{27}$$

We define elastic medium as a medium where the density of strain energy depends only on the deformation, i.e. $\mathcal{U} = \mathcal{U}(\mathbf{A}, \mathbf{K})$.

Formula (26) allows us to get correct nonlinear constitutive equations of an elastic polar medium [6]:

$$\boldsymbol{\tau} = \rho \overset{\circ}{\nabla}\mathbf{R}^{\top} \cdot \frac{\partial \mathcal{U}}{\partial \mathbf{A}} \cdot \mathbf{P}^{\top}, \tag{28}$$

$$\boldsymbol{\mu} = \rho \overset{\circ}{\nabla}\mathbf{R}^{\top} \cdot \frac{\partial \mathcal{U}}{\partial \mathbf{K}} \cdot \mathbf{P}^{\top}, \tag{29}$$

where $\mathcal{U} = \mathcal{U}(\mathbf{A}, \mathbf{K})$.

2.3.2 Nonlinear constitutive equations for Kelvin's medium

Now let us take into account that Kelvin's medium is a medium of a special kind. In this case strain energy $\mathcal{U}$ is not a function of general kind in $\mathbf{A}$ and $\mathbf{K}$, since we assume that $\mathcal{U}(\mathbf{R})$ does not depend on φ or $\nabla\varphi$. Let us search for functions in $\mathbf{A}$ and $\mathbf{K}$ (strain tensors) such that we satisfy these restrictions whenever $\mathcal{U}$ depends only on these tensors. We shall use mathematical methods that one can find in [9].

The equation

$$\frac{\partial \mathcal{U}}{\partial \varphi} = 0 \tag{30}$$

can be rewritten as

$$\left(\frac{\partial \mathcal{U}}{\partial \mathbf{A}}\right)^{\top} \cdot\cdot (\mathbf{A} \times \mathbf{m}_0) + \left(\frac{\partial \mathcal{U}}{\partial \mathbf{K}}\right)^{\top} \cdot\cdot (\mathbf{K} \times \mathbf{m}_0) = 0. \tag{31}$$

Characteristic equations for (31) are

$$\frac{\partial \mathbf{A}}{\partial \varphi} = \mathbf{A} \times \mathbf{m}_0, \qquad \frac{\partial \mathbf{K}}{\partial \varphi} = \mathbf{K} \times \mathbf{m}_0. \tag{32}$$

Density of strain energy $\mathcal{U}$ is a function of first integrals of (32). These integrals are strain tensors for medium under consideration. In the shell theory (P.A. Zhilin, [6]), the above system of equations occurs because own rotation of a shell fibre must not influence the energy of deformation. In case of a shell, the system has order 12 because we consider a shell to be a 2D object. In case of a 3D continuum, the system has order 18.

There are various possibilities in choosing the set of first integrals of (32), i.e. strain tensors of Kelvin's medium:

1. This set of functions includes all first integrals of (32):

$$\begin{aligned}
&\boldsymbol{\mathcal{E}} = (\mathbf{A} \cdot \mathbf{A}^{\top} - \mathbf{E})/2 = (\overset{\circ}{\nabla}\mathbf{R} \cdot \overset{\circ}{\nabla}\mathbf{R}^{\top} - \mathbf{E})/2, \\
&\mathbf{F} = \mathbf{K} \cdot \tilde{\mathbf{a}} \cdot \mathbf{A}^{\top} = (\overset{\circ}{\nabla}\psi \otimes \mathbf{m}_0 + \overset{\circ}{\nabla}\vartheta \otimes \mathbf{l}) \cdot \mathbf{P} \cdot \tilde{\mathbf{a}} \cdot \mathbf{P}^{\top} \cdot \overset{\circ}{\nabla}\mathbf{R}^{\top}, \\
&\boldsymbol{\gamma} = \mathbf{A} \cdot \mathbf{m}_0 = \overset{\circ}{\nabla}\mathbf{R} \cdot \mathbf{m}, \\
&\boldsymbol{\xi} = \mathbf{K} \cdot \mathbf{m}_0 \overset{(11)}{=} \overset{\circ}{\nabla}\psi \cos\vartheta + \overset{\circ}{\nabla}\varphi,
\end{aligned} \tag{33}$$

where $\mathbf{l} = \mathbf{P} \cdot \mathbf{l}_0$ and we can choose either $\tilde{\mathbf{a}} = \mathbf{E} - \mathbf{m}_0 \otimes \mathbf{m}_0$ or $\tilde{\mathbf{a}} = \mathbf{E} \times \mathbf{m}_0$. We see that $\boldsymbol{\mathcal{E}}$ is Cauchy–Green strain tensor. Tensor $\mathbf{F}$ corresponds to the "mixed" translational-angular strain.

We see that in (33) only vector $\boldsymbol{\xi}$ depends on $\nabla\varphi$. Thus we conclude that $\mathcal{U}$ does not depend on $\boldsymbol{\xi}$.

From (28), (29) we get the corresponding constitutive equations:

$$\begin{aligned}
&\boldsymbol{\tau}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\left(\frac{\partial\mathcal{U}}{\partial\boldsymbol{\mathcal{E}}}\cdot\mathbf{A}+\frac{\partial\mathcal{U}}{\partial\boldsymbol{\gamma}}\otimes\mathbf{m}_0+\left(\frac{\partial\mathcal{U}}{\partial\mathbf{F}}\right)^{\top}\cdot\mathbf{K}\cdot\tilde{\mathbf{a}}^{\top}\right)\cdot\mathbf{P}^{\top},\\
&\boldsymbol{\mu}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\,\frac{\partial\mathcal{U}}{\partial\mathbf{F}}\cdot\mathbf{A}\cdot\tilde{\mathbf{a}}^{\top}\cdot\mathbf{P}^{\top},\\
&\mathcal{U}=\mathcal{U}(\boldsymbol{\mathcal{E}},\mathbf{F},\boldsymbol{\gamma}).
\end{aligned}\tag{34}$$

2. One can suggest another set of first integrals for (32):

$$\begin{aligned}
&\boldsymbol{\Phi}=\mathbf{K}\cdot\mathbf{a}\cdot\mathbf{K}^{\top}=\sin^2\vartheta\,\overset{\circ}{\nabla}\psi\otimes\overset{\circ}{\nabla}\psi+\overset{\circ}{\nabla}\vartheta\otimes\overset{\circ}{\nabla}\vartheta,\\
&\boldsymbol{\mathcal{E}},\quad \alpha=\mathbf{m}_0\cdot\mathbf{F}\cdot\mathbf{m}_0,\quad \boldsymbol{\gamma},\quad \boldsymbol{\xi},
\end{aligned}\tag{35}$$

where $\mathbf{a}=\mathbf{E}-\mathbf{m}_0\otimes\mathbf{m}_0$. Here $\boldsymbol{\Phi}$ is responsible for angular deformations (like $\boldsymbol{\mathcal{E}}$ for translational strain), and α corresponds to the "mixed" kind of deformation.

Omitting $\boldsymbol{\xi}$ from this set for the reason mentioned above, we obtain the following constitutive equations:

$$\begin{aligned}
&\boldsymbol{\tau}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\left(\frac{\partial\mathcal{U}}{\partial\boldsymbol{\mathcal{E}}}\cdot\mathbf{A}+\frac{\partial\mathcal{U}}{\partial\boldsymbol{\gamma}}\otimes\mathbf{m}_0+\frac{\partial\mathcal{U}}{\partial\alpha}\,\mathbf{m}_0\otimes\mathbf{m}_0\cdot\mathbf{K}\cdot\mathbf{a}\right)\cdot\mathbf{P}^{\top},\\
&\boldsymbol{\mu}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\left(2\frac{\partial\mathcal{U}}{\partial\boldsymbol{\Phi}}\cdot\mathbf{K}\cdot\mathbf{a}+\frac{\partial\mathcal{U}}{\partial\alpha}\,\mathbf{m}_0\otimes\mathbf{m}_0\cdot\mathbf{A}\cdot\mathbf{a}\right)\cdot\mathbf{P}^{\top},\\
&\mathcal{U}=\mathcal{U}(\boldsymbol{\mathcal{E}},\boldsymbol{\Phi},\boldsymbol{\gamma},\alpha).
\end{aligned}\tag{36}$$

Of course these two variants are not the only ones possible; in fact, there is an infinite amount of sets of first integrals of (32).

Set (33) is a set of independent integrals of (32) in the case of shells. In the case under consideration, (33) as well as (35) include all independent integrals and some dependent ones. There are many ways of eliminating dependent functions. For example, this is a set of independent integrals:

$$\boldsymbol{\mathcal{E}}_1=\boldsymbol{\mathcal{E}}-\boldsymbol{\mathcal{E}}\cdot\cdot\,\mathbf{m}_0\otimes\mathbf{m}_0\otimes\mathbf{m}_0\otimes\mathbf{m}_0,\quad \mathbf{F}\cdot\mathbf{a},\quad \boldsymbol{\gamma},\quad \boldsymbol{\xi}.\tag{37}$$

We also can consider another set of independent integrals:

$$\boldsymbol{\mathcal{E}}_1,\quad \boldsymbol{\Phi}_1=\boldsymbol{\Phi}-\boldsymbol{\Phi}\cdot\cdot\,\mathbf{m}_0\otimes\mathbf{m}_0\otimes\mathbf{m}_0\otimes\mathbf{m}_0,\quad \alpha,\quad \boldsymbol{\gamma},\quad \boldsymbol{\xi}.\tag{38}$$

The density of strain energy $\mathcal{U}$ depends only on these functions. Excluding $\boldsymbol{\xi}$ from these sets we have $\mathcal{U}=\mathcal{U}(\boldsymbol{\mathcal{E}}_1,\mathbf{F}\cdot\mathbf{a},\boldsymbol{\gamma})$ or $\mathcal{U}=\mathcal{U}(\boldsymbol{\mathcal{E}}_1,\boldsymbol{\Phi}_1,\boldsymbol{\gamma},\alpha)$.

Corresponding to (37) and (38), the constitutive equations are:

$$\begin{aligned}
&\boldsymbol{\tau}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\left(\frac{\partial\mathcal{U}}{\partial\boldsymbol{\mathcal{E}}_1}\cdot\mathbf{A}+\frac{\partial\mathcal{U}}{\partial\boldsymbol{\gamma}}\otimes\mathbf{m}_0+\left(\frac{\partial\mathcal{U}}{\partial\mathbf{F}\cdot\mathbf{a}}\right)^{\top}\cdot\mathbf{K}\cdot\tilde{\mathbf{a}}^{\top}\right)\cdot\mathbf{P}^{\top},\\
&\boldsymbol{\mu}=\overset{\circ}{\nabla}\mathbf{R}^{\top}\cdot\rho\,\frac{\partial\mathcal{U}}{\partial\mathbf{F}\cdot\mathbf{a}}\cdot\mathbf{a}\cdot\mathbf{A}\cdot\tilde{\mathbf{a}}^{\top}\cdot\mathbf{P}^{\top},\\
&\mathcal{U}=\mathcal{U}(\boldsymbol{\mathcal{E}}_1,\mathbf{F}\cdot\mathbf{a},\boldsymbol{\gamma})
\end{aligned}\tag{39}$$

and

$$\begin{aligned}
&\boldsymbol{\tau} = \overset{\circ}{\nabla}\mathbf{R}^\top \cdot \rho \left(\frac{\partial \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}_1} \cdot \mathbf{A} + \frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}} \otimes \mathbf{m}_0 + \frac{\partial \mathcal{U}}{\partial \alpha} \mathbf{m}_0 \otimes \mathbf{m}_0 \cdot \mathbf{K} \cdot \mathbf{a} \right) \cdot \mathbf{P}^\top, \\
&\boldsymbol{\mu} = \overset{\circ}{\nabla}\mathbf{R}^\top \cdot \rho \left(2 \frac{\partial \mathcal{U}}{\partial \boldsymbol{\Phi}_1} \cdot \mathbf{K} \cdot \mathbf{a} + \frac{\partial \mathcal{U}}{\partial \alpha} \mathbf{m}_0 \otimes \mathbf{m}_0 \cdot \mathbf{A} \cdot \mathbf{a} \right) \cdot \mathbf{P}^\top, \\
&\mathcal{U} = \mathcal{U}(\boldsymbol{\mathcal{E}}_1, \boldsymbol{\Phi}_1, \boldsymbol{\gamma}, \alpha).
\end{aligned} \tag{40}$$

respectively.

It is possible to use any of (33), (35), (37), (38) as strain tensors. If we use (33) or (35), twice or more do we take into account dependence $\mathcal{U}$ on certain kinds of strain. If we use (37) or (38) and consider the simplest nonlinear theory (taking $\mathcal{U}$ to be the quadratic form of the strain tensors), $\mathcal{U}$ will depend on chosen strain tensors (37) or (38) in a simple way, and on other (dependent) strain tensors in a complicated way. The latter seems to be less convenient.

NB: Set of the functions (33) as well as (35) includes all independent kinds of deformation that induce stresses in Kelvin's medium. If $\mathcal{U}$ depends of any other kind of deformation, the latter can be expressed as a function of strain tensors (33) or (35). If we omit α from (33), this set will not contain all independent strain tensors. Tensor $\mathbf{F}$ in (33) and α in (35) are special "mixed" kinds of deformation that depend on the product of the translational displacements gradient and the gradient of a turn-tensor of a point body. If $\mathcal{U}$ depends on this kind of deformation, this is sufficient for existence of a coupling between translational and angular displacements.

2.3.3 Restrictions on the stress and couple tensors for Kelvin's medium

The density of strain energy in Kelvin's medium has to satisfy restrictions $\dfrac{\partial \mathcal{U}}{\partial \varphi} = 0$ and $\dfrac{\partial \mathcal{U}}{\partial \nabla \varphi} = 0$. In the subsection above we have rewritten these restrictions in terms of strain energy and strain tensors. Now let us get another form for them in terms of stress tensors.

The fact that internal stresses in Kelvin's medium can not be induced by a gradient of own rotation of its particles having axial symmetry can be rewritten as $\boldsymbol{\mu} \cdot \mathbf{m} = \mathbf{0}$. Indeed, using (29) we may write

$$\begin{aligned}
\boldsymbol{\mu} \cdot \mathbf{m} &= \rho\, \overset{\circ}{\nabla}\mathbf{R}^\top \cdot \frac{\partial \mathcal{U}}{\partial \mathbf{K}} \cdot \mathbf{P}^\top \cdot \mathbf{m} = \rho\, \overset{\circ}{\nabla}\mathbf{R}^\top \cdot \frac{\partial \mathcal{U}}{\partial \mathbf{K}} \cdot \mathbf{m}_0 = \\
&= \rho\, \overset{\circ}{\nabla}\mathbf{R}^\top \cdot \frac{\partial \mathcal{U}}{\partial \mathbf{K} \cdot \mathbf{m}_0} = \rho\, \overset{\circ}{\nabla}\mathbf{R}^\top \cdot \frac{\partial \mathcal{U}}{\partial \boldsymbol{\xi}} = \mathbf{0},
\end{aligned} \tag{41}$$

since $\boldsymbol{\xi} = \overset{\circ}{\nabla}\psi \cos\theta + \overset{\circ}{\nabla}\varphi$. Thus we see that

$$\frac{\partial \mathcal{U}}{\partial \nabla \varphi} = 0 \quad \Longleftrightarrow \quad \boldsymbol{\mu} \cdot \mathbf{m} = \mathbf{0}. \tag{42}$$

Our assumption $\dfrac{\partial \mathcal{U}}{\partial \varphi} = 0$ involves the analog to the "6th balance equation" in the theory of shells [6]:

$$\boldsymbol{\tau}_\times \cdot \mathbf{m} = \boldsymbol{\mu}^\top \cdot\cdot \nabla \mathbf{m}. \tag{43}$$

This can be got from (34) or (36), but it is more easy to get it directly from (28), (29). Let us transform the left-hand side of (43):

$$\boldsymbol{\tau}_\times \cdot \mathbf{m} \overset{(28)}{=} -\rho\,(\mathbf{P}\cdot \left(\frac{\partial \mathcal{U}}{\partial \mathbf{A}}\right)^{\top} \cdot \overset{\circ}{\nabla}\mathbf{R}) \cdot\cdot\, (\mathbf{E}\times\mathbf{m}) = -\rho \left(\frac{\partial \mathcal{U}}{\partial \mathbf{A}}\right)^{\top} \cdot\cdot\, ((\overset{\circ}{\nabla}\mathbf{R}\times\mathbf{m})\cdot\mathbf{P}) =$$
$$= -\rho \left(\frac{\partial \mathcal{U}}{\partial \mathbf{A}}\right)^{\top} \cdot\cdot\, ((\overset{\circ}{\nabla}\mathbf{R}\cdot\mathbf{P})\times(\mathbf{P}^{\top}\cdot\mathbf{m})) \overset{(10)}{=} -\rho \left(\frac{\partial \mathcal{U}}{\partial \mathbf{A}}\right)^{\top} \cdot\cdot\, (\mathbf{A}\times\mathbf{m}_0). \quad (44)$$

Now we shall transform the right-hand side of (43):

$$\boldsymbol{\mu}^{\top} \cdot\cdot\, \nabla\mathbf{m} \overset{(29)}{=} \rho\,(\mathbf{P}\cdot \left(\frac{\partial \mathcal{U}}{\partial \mathbf{K}}\right)^{\top} \cdot \overset{\circ}{\nabla}\mathbf{R}) \cdot\cdot\, \nabla\mathbf{m} \overset{(5)}{=} \rho\,(\mathbf{P}\cdot \left(\frac{\partial \mathcal{U}}{\partial \mathbf{K}}\right)^{\top} \cdot \mathbf{r}^s\otimes\mathbf{R}_s) \cdot\cdot\, (\mathbf{R}^i\otimes\boldsymbol{\Phi}_i\times\mathbf{m}) =$$
$$= \rho \left(\frac{\partial \mathcal{U}}{\partial \mathbf{K}}\right)^{\top} \cdot\cdot\, (\mathbf{r}^i\otimes(\boldsymbol{\Phi}_i\times\mathbf{m})\cdot\mathbf{P}) = \rho \left(\frac{\partial \mathcal{U}}{\partial \mathbf{K}}\right)^{\top} \cdot\cdot\, (\mathbf{r}_i\otimes(\mathbf{P}^{\top}\cdot\boldsymbol{\Phi}_i)\times(\mathbf{P}^{\top}\cdot\mathbf{m})) =$$
$$= \rho \left(\frac{\partial \mathcal{U}}{\partial \mathbf{K}}\right)^{\top} \cdot\cdot\, ((\mathbf{r}^i\otimes\boldsymbol{\Phi}_i\cdot\mathbf{P})\times\mathbf{m}_0) \overset{(10)}{=} \rho \left(\frac{\partial \mathcal{U}}{\partial \mathbf{K}}\right)^{\top} \cdot\cdot\, (\mathbf{K}\times\mathbf{m}_0). \quad (45)$$

Thus we have

$$\boldsymbol{\mu}^{\top} \cdot\cdot\, \nabla\mathbf{m} - \boldsymbol{\tau}_\times\cdot\mathbf{m} = \rho \left(\frac{\partial \mathcal{U}}{\partial \mathbf{K}}\right)^{\top} \cdot\cdot\, (\mathbf{K}\times\mathbf{m}_0) + \rho \left(\frac{\partial \mathcal{U}}{\partial \mathbf{A}}\right)^{\top} \cdot\cdot\, (\mathbf{A}\times\mathbf{m}_0) =$$
$$= \rho \left(\frac{\partial \mathcal{U}}{\partial \mathbf{K}}\right)^{\top} \cdot\cdot\, \frac{\partial \mathbf{K}}{\partial \varphi} + \rho \left(\frac{\partial \mathcal{U}}{\partial \mathbf{A}}\right)^{\top} \cdot\cdot\, \frac{\partial \mathbf{A}}{\partial \varphi} = \rho\,\frac{\partial \mathcal{U}}{\partial \varphi}, \quad (46)$$

and we may conclude that

$$\frac{\partial \mathcal{U}}{\partial \varphi} = 0 \quad \Longleftrightarrow \quad \boldsymbol{\tau}_\times\cdot\mathbf{m} = \boldsymbol{\mu}^{\top} \cdot\cdot\, \nabla\mathbf{m}. \quad (47)$$

At the same time in general case $\boldsymbol{\tau}_\times\cdot\mathbf{m} \neq 0$.

If $\mathcal{U} = \mathcal{U}(\boldsymbol{\mathcal{E}}, \boldsymbol{\Phi}, \boldsymbol{\gamma})$, i.e. $\mathcal{U}$ does not depend on α, formula (36) yields

$$\boldsymbol{\tau}_\times\cdot\mathbf{m} = \boldsymbol{\mu}^{\top} \cdot\cdot\, \nabla\mathbf{m} = 0, \quad (48)$$

but if we assume this, we lose the dependence on one of the kinds of deformations that can exist and is not forbidden by thermodynamics.

Later it will be shown that (48) is valid in the linear theory. It means that the linear theory can not take into account dependence of strain energy on all kinds of "mixed" deformation, i.e. describe completely an interaction between angular and translational subsystems.

NB: The angular velocity of own rotation $\dot{\varphi}$ can not be changed by any internal forces or moments since they do not perform mechanical work on own rotation of the body. If external body moment has no projection on the axis $\mathbf{m}$ of a point body, $\dot{\varphi}$ does not depend on time and can be considered a constant of the medium.

2.4 Linear constitutive equations

Let us assume that in the initial configuration stresses are equal to zero, and that $\overset{\circ}{\nabla}\mathbf{m}_0 = \mathbf{0}$. Let angles of nutation and translational displacements to be infinitesimal, i.e.

$$\mathbf{P} \approx (\mathbf{E} + \theta \times \mathbf{E}) \cdot \mathbf{P}_1(\varphi \mathbf{m}_0), \;\; \theta = o(1), \quad \mathbf{u} = \mathbf{R} - \mathbf{r} = o(1). \tag{49}$$

It is possible to obtain the linear theory by different ways. The simplest one is to expand the law of energy balance (25) and to require independence $\rho\dot{\mathcal{U}}$ on the angular velocity of own rotation $\dot{\varphi}$. After that one will obtain the linear analog for (26) and linearized restrictions (47), (42). It gives the possibility to get linear constitutive equations. It was done in [10].

We shall obtain the linear theory from the nonlinear one. We shall use notation $[\cdot]_n$ the term of order n in $\mathbf{u}, \theta$. One can see that $[\mathbf{A}]_0 = \mathbf{P}_1(\varphi\mathbf{m}_0), [\mathbf{K}]_0 = \mathbf{0}$, and

$$[\mathbf{A}]_1 = \mathbf{g} \cdot \mathbf{P}_1(\varphi\mathbf{m}_0), \;\; [\mathbf{K}]_1 = \mathbf{f} \cdot \mathbf{P}_1(\varphi\mathbf{m}_0), \;\; \mathbf{g} = \overset{\circ}{\nabla}\mathbf{u} + \theta \times \mathbf{E}, \;\; \mathbf{f} = \overset{\circ}{\nabla}\theta. \tag{50}$$

We shall expand nonlinear constitutive equations (34) with $\tilde{\mathbf{a}} = \mathbf{a}$.

$$[\boldsymbol{\tau}]_0 = \rho_0 \left(\left[\frac{\partial \mathcal{U}}{\partial \mathcal{E}}\right]_0 + \left[\frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}}\right]_0 \otimes \mathbf{m}_0 \right), \tag{51}$$

$$[\boldsymbol{\mu}]_0 = \rho_0 \left[\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right]_0 \cdot \mathbf{a}. \tag{52}$$

We assume that internal stresses are equal to zero in the initial configuration. Thus we have to require $[\boldsymbol{\tau}]_0 = \mathbf{0}$, $[\boldsymbol{\mu}]_0 = \mathbf{0}$ and we conclude that

$$\left[\frac{\partial \mathcal{U}}{\partial \mathcal{E}}\right]_0 + \left[\frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}}\right]_0 \otimes \mathbf{m}_0 = \mathbf{0}, \quad \left[\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right]_0 \cdot \mathbf{a} = \mathbf{0}. \tag{53}$$

Taking (53), (50) into account we continue to expand (34):

$$[\boldsymbol{\tau}]_1 = \rho_0 \left(\left[\frac{\partial \mathcal{U}}{\partial \mathcal{E}}\right]_1 + \left[\frac{\partial \mathcal{U}}{\partial \mathcal{E}}\right]_0 \cdot \mathbf{g} + \left[\frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}}\right]_1 \otimes \mathbf{m}_0 + \left[\left(\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right)^{\top}\right]_0 \cdot \mathbf{f} \cdot \mathbf{a} \right), \tag{54}$$

$$[\boldsymbol{\mu}]_1 = \rho_0 \left(\left[\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right]_1 + \left[\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right]_0 \cdot \mathbf{g} \right) \cdot \mathbf{a}. \tag{55}$$

Assuming $\mathcal{U}$ to be sufficiently smooth in a neighbourhood of initial configuration, having done some calculations we obtain linear constitutive equations:

$$\begin{aligned} \boldsymbol{\tau} &= ({}^4\mathbf{X} \cdot\cdot\, \mathbf{g} + {}^4\mathbf{Y} \cdot\cdot\, \mathbf{f})^{\top}, \\ \boldsymbol{\mu} &= (\mathbf{g} \cdot\cdot\, {}^4\mathbf{Y} + {}^4\mathbf{Z} \cdot\cdot\, \mathbf{f})^{\top}, \end{aligned} \tag{56}$$

Here ${}^4\mathbf{X} = X_{mnkl}\mathbf{r}_m\mathbf{r}_n\mathbf{r}_k\mathbf{r}_l, {}^4\mathbf{Y} = Y_{mn\alpha l}\mathbf{r}_m\mathbf{r}_n\mathbf{r}_\alpha\mathbf{r}_l$, and ${}^4\mathbf{Z} = Z_{\alpha m\beta l}\mathbf{r}_\alpha\mathbf{r}_m\mathbf{r}_\beta\mathbf{r}_l$ are tensors of elastic constants:

$$
\begin{aligned}
{}^4\mathbf{X} = &\left[\frac{\partial^2 \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}^2}\right]_0 + \left[\frac{\partial}{\partial \boldsymbol{\mathcal{E}}} \otimes (\mathbf{m}_0 \frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}})\right]_0 + \mathbf{m}_0 \left[\frac{\partial}{\partial \boldsymbol{\gamma}} \frac{\partial \mathcal{U}}{\partial \boldsymbol{\mathcal{E}}}\right]_0 + \mathbf{m}_0 \left[\frac{\partial}{\partial \boldsymbol{\gamma}} (\mathbf{m}_0 \otimes \frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}})\right]_0 + \\
&+ e_0 \mathbf{r}_k \otimes \mathbf{m}_0 \otimes \mathbf{r}_k \otimes \mathbf{m}_0, \\
{}^4\mathbf{Y} = &\left[\frac{\partial^2 \mathcal{U}}{\partial \varepsilon_{mn} \partial F_{k\alpha}}\right]_0 \mathbf{r}_m \otimes \mathbf{r}_n \otimes \mathbf{r}_\alpha \otimes \mathbf{r}_k + \mathbf{m}_0 \otimes \left[\frac{\partial^2 \mathcal{U}}{\partial \gamma_s \partial F_{k\alpha}}\right]_0 \mathbf{r}_s \otimes \mathbf{r}_\alpha \otimes \mathbf{r}_k + \\
&+ \mathbf{r}_\alpha \otimes \mathbf{m}_0 \otimes \mathbf{r}_\alpha \otimes \mathbf{f}_0, \\
{}^4\mathbf{Z} = &\left[\frac{\partial^2 \mathcal{U}}{\partial F_{m\alpha} \partial F_{k\beta}}\right]_0 \mathbf{r}_\alpha \otimes \mathbf{r}_m \otimes \mathbf{r}_\beta \otimes \mathbf{r}_k,
\end{aligned}
\tag{57}
$$

where $\left[\frac{\partial \mathcal{U}}{\partial \boldsymbol{\gamma}}\right]_0 = -e_0 \mathbf{m}_0$, $\left[\frac{\partial \mathcal{U}}{\partial \mathbf{F}}\right]_0 = \mathbf{f}_0 \otimes \mathbf{m}_0$ (this can be obtained from (53) taking into account $\boldsymbol{\mathcal{E}} = \boldsymbol{\mathcal{E}}^\top$).

The linear approximation for 6th balance equation (47) coincides with linearized (48), because we assume that in the initial configuration $\overset{\circ}{\nabla} \mathbf{m}_0 = \mathbf{0}$, and hence the right-hand side of nonlinear 6th balance equation (43) is equal to zero in the linear approximation. Thus we see that linearized restrictions (47), (42) look as

$$
[\boldsymbol{\tau}_\times]_1 \cdot \mathbf{m}_0 = 0, \quad [\boldsymbol{\mu}]_1 \cdot \mathbf{m}_0 = 0. \tag{58}
$$

The restrictions on the tensors of elastic moduli following from (58) are:

$$
\begin{aligned}
\mathbf{m}_0 \cdot (\boldsymbol{\epsilon} \cdot\cdot {}^4\mathbf{X}) = \mathbf{0}, \\
\mathbf{m}_0 \cdot (\boldsymbol{\epsilon} \cdot\cdot {}^4\mathbf{Y}) = \mathbf{0},
\end{aligned}
\tag{59}
$$

where $\boldsymbol{\epsilon} = -\mathbf{E} \times \mathbf{E}$. One can see that tensors ${}^4\mathbf{X}, {}^4\mathbf{Y}, {}^4\mathbf{Z}$ satisfy (59), and there are no other restrictions for their components.

Thus we can state that stress and couple tensors in the linear theory are determined by

$$
[\boldsymbol{\tau}]_1 = \rho_0 \frac{\partial \mathcal{U}}{\partial \mathbf{g}}, \quad [\boldsymbol{\mu}]_1 = \rho_0 \frac{\partial \mathcal{U}}{\partial \mathbf{f}}, \tag{60}
$$

where the expression for strain energy in the linear theory is

$$
\begin{aligned}
\rho_0 \mathcal{U} = &\frac{1}{2} \mathbf{g} \cdot\cdot {}^4\mathbf{X} \cdot\cdot \mathbf{g} + \mathbf{g} \cdot\cdot {}^4\mathbf{Y} \cdot\cdot \mathbf{f} + \frac{1}{2} \mathbf{f} \cdot\cdot {}^4\mathbf{Z} \cdot\cdot \mathbf{f} = \\
= &\frac{1}{2} \left(\mathbf{g}^S \cdot\cdot {}^4\mathbf{T} \cdot\cdot \mathbf{g}^S + \mathbf{g}^A \cdot\cdot {}^4\mathbf{U} \cdot\cdot \mathbf{g}^A\right) + \mathbf{g}^S \cdot\cdot {}^4\mathbf{W} \cdot\cdot \mathbf{g}^A + \\
&+ \mathbf{g}^S \cdot\cdot {}^4\mathbf{H} \cdot\cdot \mathbf{f} + \mathbf{g}^A \cdot\cdot {}^4\mathbf{N} \cdot\cdot \mathbf{f} + \frac{1}{2} \mathbf{f} \cdot\cdot {}^4\mathbf{Z} \cdot\cdot \mathbf{f}.
\end{aligned}
\tag{61}
$$

Here $\mathbf{T}, \mathbf{U}, \mathbf{W}, \mathbf{H}, \mathbf{N}$ are tensors of elastic constants which are convenient to use. Tensors $\mathbf{Y} = \mathbf{H} + \mathbf{N}$, $\mathbf{U}$, and $\mathbf{W}$ are responsible for the coupling between angular and translational displacements.

We can find expressions (58)–(61) in [10], where the linear theory of medium under consideration is proposed.

From the formal point of view one can linearize (36) as well as (34) or any other variants of nonlinear constitutive equations obtained by the method described above for different systems of strain tensors. However it is much more simple to do for (34) or (39), because the linear terms of strain tensors (33) and (37) are not equal to zero and it is possible to go from (54), (55) to (56), (57).

2.5 Linear dynamic equations

The law of balance of momentum looks the same way as (17). To write down the law of balance of kinetic moment we must linearize the right-hand side of equation (19). Here $\boldsymbol{\tau}, \boldsymbol{\mu}$ are stress and couple tensors determined by (56). Using formulae (56), we get linear dynamic equations in displacements:

$$\begin{aligned}
&\overset{\circ}{\nabla} \cdot ({}^4\mathbf{X} \cdot\cdot\, \mathbf{g} + {}^4\mathbf{Y} \cdot\cdot\, \mathbf{f})^\top + \rho\,\mathbf{Q} = \rho\,\ddot{\mathbf{u}}, \\
&\overset{\circ}{\nabla} \cdot (\mathbf{g} \cdot\cdot\, {}^4\mathbf{Y} + {}^4\mathbf{Z} \cdot\cdot\, \mathbf{f})^\top + ({}^4\mathbf{X} \cdot\cdot\, \mathbf{g} + {}^4\mathbf{Y} \cdot\cdot\, \mathbf{f})^\top_\times + \rho\,\mathbf{L} = \rho\,(\mu\,\ddot{\boldsymbol{\theta}} + \lambda\,\dot{\varphi}\,\dot{\boldsymbol{\theta}} \times \mathbf{m}_0).
\end{aligned} \qquad (62)$$

3 The analogy between Kelvin's medium, shells and ferromagnets

3.1 Elastic shells and Kelvin's medium

One may consider a non-classical elastic shell as a material surface every point of which is a "fibre" that can turn and move. Thus a shell is a 2D polar medium. Constitutive equations for shells can be obtained via the method described in [6]. As the own rotation of a fibre can not induce any stresses in the shell, the way to obtain constitutive equations of Kelvin's medium is exactly the same. Hence we can find an analogy between constitutive equations. In particular, for elastic shells one may use strain tensors (33) and constitutive equations (34), where ∇ and $\overset{\circ}{\nabla}$ are 2D nabla operators. It is more expedient to use instead of $\boldsymbol{\mathcal{E}}$ its 2D analog $(\overset{\circ}{\nabla}\mathbf{R} \cdot (\overset{\circ}{\nabla}\mathbf{R})^\top - \mathbf{a})/2$ in case of shells. Set of strain tensors (33) for shells describes deformations in a simpler way than in Kelvin's medium: in 2D case all components of strain tensors (33) are independent, and all independent kinds of deformation can be expressed as functions in these components, and in 3D case components of strain tensors (33) include all independent kinds of deformation, but some of these components are functions of others.

There are two differences between a shell and Kelvin's medium: 1) a shell is a 2D surface and Kelvin's medium is a 3D continuum; 2) every point body of Kelvin's medium (unlike a fibre of a shell) has a finite or large angular velocity of own rotation and non-zero axial moment of inertia λ. The second difference is essential when considering wave processes but does not influence the constitutive equations.

3.2 Ferromagnets and Kelvin's medium

3.2.1 Nonlinear equations

One example of deformable solids is saturated elastic ferromagnetic insulators. Below we will state some facts about ferromagnets which can be found in [7]. We will consider insulators to avoid necessity to take into account electric fields. We will not consider any processes concerning heat transfer though this restriction is not essential.

Every point of such a ferromagnet is characterized by displacement $\mathbf{u}$ and by density of a vector of magnetic moment $\mathbf{S}$. The state of magnetic saturation is defined as the state of ferromagnet when $|\mathbf{S}|$ is constant in radius-vector $\mathbf{R}$ and time t.

Forces and moments both of elastic and quantum mechanical nature act upon every point of a ferromagnet. The density of a moment induced by an external magnetic induction $\mathbf{B}^e$ is equal to

$$\mathbf{L} = \mathbf{S} \times \mathbf{B}^e. \tag{63}$$

Exchange interaction (interaction between spins depending on their relative turn) results in a close-range moment interaction; density of the above moment acting upon an elementary surface with the normal $\mathbf{n}$ is equal to

$$\mathbf{M}^{exc}_{(\mathbf{n})} = \mathbf{S} \times \mathbf{\Gamma}_{(\mathbf{n})} \tag{64}$$

where $\mathbf{\Gamma}_{(\mathbf{n})}$ is so called "contact exchange force". This allows to consider a tensor of exchange interactions $\mathcal{B}$ such that

$$\mathbf{S} \times (\rho\, \mathbf{\Gamma}_{(\mathbf{n})} - \mathbf{n} \cdot \mathcal{B}) = \mathbf{0}. \tag{65}$$

The power of a contact exchange force is equal to $\rho\, \mathbf{\Gamma}_{(\mathbf{n})} \cdot \dot{\mathbf{S}}$. Apart from the moment of exchange interaction, there exist forces of elastic nature and a "spin-lattice" moment. The latter depends on the direction of a magnetic moment of a point body relatively to the lattice.

The own kinetic moment of a point body in ferromagnet is equal to $\rho\, \mathbf{S}/\gamma$, where γ is the gyromagnetic ratio (constant).

The law of balance of force for elastic ferromagnetic insulator is

$$\frac{d}{dt} \int\limits_{\triangle V} \rho\, \mathbf{v}\, dV = \int\limits_{\triangle V} \rho\, \mathbf{Q}\, dV + \int\limits_{\Sigma} \boldsymbol{\tau}_{(\mathbf{n})}\, d\Sigma. \tag{66}$$

The law of balance of moment for elastic ferromagnetic insulator is

$$\frac{d}{dt} \int\limits_{\triangle V} \rho\, (\mathbf{r} \times \mathbf{v} + \mathbf{S}/\gamma)\, dV = \int\limits_{\triangle V} \rho\, (\mathbf{L} + \mathbf{R} \times \mathbf{Q})\, dV + \int\limits_{\Sigma} (\rho\, \mathbf{S} \times \mathbf{\Gamma}_{(\mathbf{n})} + \mathbf{R} \times \boldsymbol{\tau}_{(\mathbf{n})})\, d\Sigma. \tag{67}$$

The law of energy balance looks in the following way:

$$\frac{d}{dt} \int\limits_{\triangle V} \rho\, (\mathbf{v} \cdot \mathbf{v}/2 + \mathcal{U} - \mathbf{S} \cdot \mathbf{B}^e)\, dV =$$

$$= \int\limits_{\triangle V} \rho\, (\mathbf{Q} \cdot \mathbf{v} - \mathbf{S} \cdot \dot{\mathbf{B}}^e)\, dV + \int\limits_{\Sigma} (\rho\, \mathbf{\Gamma}_{(\mathbf{n})} \cdot \dot{\mathbf{S}} + \boldsymbol{\tau}_{(\mathbf{n})} \cdot \mathbf{v})\, d\Sigma. \tag{68}$$

For saturated ferromagnet $|\mathbf{S}| = \text{const}$, and one can write

$$\dot{\mathbf{S}} = \boldsymbol{\omega} \times \mathbf{S}. \tag{69}$$

Basing on (66)–(68) G.A. Maugin [7] obtains constitutive equations of ferromagnets using phenomenological approach.

Let us consider a saturated ferromagnet in terms of mechanics of elastic polar medium and interpret the facts stated above. It is a medium with interactions both of moment and force nature. Vector of translational displacement $\mathbf{u}$ and vector $\mathbf{S}$ are kinematic characteristics of every point body. Since $|\mathbf{S}| = \text{const}$, we may write $\mathbf{S} = \mathbf{P}(\mathbf{R}, t) \cdot \mathbf{S}_0$, where $\mathbf{S}_0$ is $\mathbf{S}$ in the initial configuration, and we can interpret $\boldsymbol{\omega}$ in equation (69) as angular velocity corresponding to the turn-tensor $\mathbf{P}$. One may use $\mathbf{P}$ as kinematic characteristic of a point body instead of $\mathbf{S}$. We may represent

$$\mathbf{P}(t) = \mathbf{P}_3(\psi \mathbf{m}_0) \cdot \mathbf{P}_2(\vartheta \mathbf{l}_0) \cdot \mathbf{P}_1(\varphi \mathbf{m}_0), \tag{70}$$

where $\mathbf{m}_0 = \mathbf{S}_0/|\mathbf{S}_0|$. Vector $\mathbf{S}$ does not depend on φ and no internal stresses can be induced by $\mathbf{P}_1$.

The power of an exchange force is $\rho\, \boldsymbol{\Gamma}_{(\mathbf{n})} \cdot \dot{\mathbf{S}} \overset{(69)}{=} \rho\, \boldsymbol{\Gamma}_{(\mathbf{n})} \cdot (\boldsymbol{\omega} \times \mathbf{S}) \overset{(64)}{=} \rho\, \mathbf{M}^{exc}_{(\mathbf{n})} \cdot \boldsymbol{\omega}$, i.e. it is equal to the power of an exchange moment performing mechanical work upon rotation of a point body with angular velocity $\boldsymbol{\omega}$. Thus, taking into account (69), (63), we rewrite the law (68) as

$$\frac{d}{dt} \int\limits_{\triangle V} \rho\, (\mathbf{v} \cdot \mathbf{v}/2 + \mathcal{U})\, dV = \\ = \int\limits_{\triangle V} \rho\, (\mathbf{Q} \cdot \mathbf{v} + \mathbf{L} \cdot \boldsymbol{\omega})\, dV + \int\limits_{\Sigma} (\rho\, \mathbf{M}^{exc}_{(\mathbf{n})} \cdot \boldsymbol{\omega} + \boldsymbol{\tau}_{(\mathbf{n})} \cdot \mathbf{v})\, d\Sigma. \tag{71}$$

Taking into account (64) – (67) and (69), we may write down the local form of (71):

$$\rho\, \dot{\mathcal{U}} = \boldsymbol{\tau} \cdot\cdot \nabla \mathbf{v} - \boldsymbol{\tau}_{\times} \cdot \boldsymbol{\omega} - (\mathcal{B} \times \mathbf{S}) \cdot\cdot \nabla \boldsymbol{\omega} \tag{72}$$

Comparing (25) and (72), we see that they coincide if we put

$$\boldsymbol{\mu} = -\mathcal{B} \times \mathbf{S} \equiv -S\, \mathcal{B} \times \mathbf{m}. \tag{73}$$

We can represent $\boldsymbol{\mu}$ in Kelvin's media in this way due to restriction (42).

We may try to draw a parallel between an elastic ferromagnetic insulator and Kelvin's medium. Let us consider medium with point bodies having infinitesimal moments of inertia with densities λ and μ but large angular velocity of own rotation $\dot{\varphi}$; vector $\mathbf{m}$ is an axis of a point body. In this case, the own kinetic moment of a particle is approximately equal to $\rho\, \lambda \dot{\varphi} \mathbf{m}$.

If the external moment has no projection on the axis of a point body $\mathbf{m}$ then $\dot{\varphi}$ is constant, and we can denote $S/\gamma = \lambda \dot{\varphi}$, $\mathbf{S} = S\mathbf{m}$. Thus we have $\rho\, \mathbf{S}/\gamma$ to be the kinetic moment of a point body, $|\mathbf{S}| = \text{const}$, and direction of $\mathbf{S}$ coincides with the axis of a particle.

Since we assume that $\mathbf{L}\cdot\mathbf{m}=0$, there exists vector $\mathbf{B}^e$ such that $\mathbf{L}=\mathbf{S}\times\mathbf{B}^e$, and $\mathbf{B}^e$ can be interpreted as external magnetic induction.

We may introduce the couple tensor $\boldsymbol{\mu}$ in a usual way. It has been shown (see section 2) that $\boldsymbol{\mu}\cdot\mathbf{m}=\mathbf{0}$. This allows us to represent $\boldsymbol{\mu}$ as $\boldsymbol{\mu}=-S\,\boldsymbol{\mathcal{B}}\times\mathbf{m}$ and to interpret $\boldsymbol{\mathcal{B}}$ as a tensor of exchange interaction.

If we rewrite the laws of balance of momentum, kinetic moment, and energy for Kelvin's medium in the new notation, we get these laws for saturated elastic ferromagnetic insulators that one can find in [7]. This allows us to obtain nonlinear constitutive equations for elastic ferromagnetic insulators with the method described above and the result will be exactly the same. Note that (33) (where $\tilde{\mathbf{a}}=\mathbf{E}\times\mathbf{m}_0$) can be found as an intermediate result in [7].

If we use (35) and suppose that $\mathcal{U}$ does not depend on α, we obtain the system of constitutive equations being used in the theory of ferromagnets. The assumption that $\dfrac{\partial\mathcal{U}}{\partial\alpha}=0$ demands that $\boldsymbol{\tau}_\times\cdot\mathbf{m}=0$, thus there exists a vector $\mathbf{B}^L$ such that $\boldsymbol{\tau}_\times=M\mathbf{B}^L\times\mathbf{m}$. In terms of the theory of ferromagnets, it means that a spin-lattice interaction is provided only by "local magnetic induction" ($\mathbf{B}^L$) acting upon spins. If we do not make the above assumption, then we have $\boldsymbol{\tau}_\times\cdot\mathbf{m}=\boldsymbol{\mu}^\top\cdot\cdot\nabla\mathbf{m}$ (see (43)). We see that when changing $\mathcal{U}(\boldsymbol{\mathcal{E}},\mathbf{F},\boldsymbol{\gamma})$ to $\mathcal{U}(\boldsymbol{\mathcal{E}},\boldsymbol{\Phi},\boldsymbol{\gamma})$ as G.A. Maugin does in [7], we lose the dependence on one of kinds of deformation (α), making the transformation dubious.

In [11] we can find that representation for $\mathcal{U}$ may include the term $\mathbf{m}\cdot(\nabla\times\mathbf{m})=\mathrm{tr}(\mathbf{K}\cdot\mathbf{a}\cdot\mathbf{A}^{-1})$. To ensure that $\boldsymbol{\tau}_\times\cdot\mathbf{m}=0$ (as G.A. Maugin [7] requires) we must set $\mathcal{U}=\mathcal{U}(\boldsymbol{\mathcal{E}},\boldsymbol{\Phi},\boldsymbol{\gamma})$, i.e. exclude dependence $\mathcal{U}$ on $\mathbf{F}$. However, it is impossible to satisfy both of these requirements, because $\mathbf{F}=\mathbf{K}\cdot\mathbf{a}\cdot\mathbf{A}^{-1}\cdot(2\boldsymbol{\mathcal{E}}-\boldsymbol{\gamma}\otimes\boldsymbol{\gamma}+\mathbf{E})$.

Thus we have an exact analogy between ferromagnets and Kelvin's medium. An axis of a point body corresponds to the unit vector of a spin, all angular characteristics correspond to magnetic subsystem and translational ones to elastic subsystem. We make the following analogies:

$\mathbf{u}$ is the translational displacement in both media;

$\mathbf{m}$ is the axis of a point body in a Kelvin's medium and the unit vector of a magnetic moment (or of a spin) in a ferromagnet;

$\boldsymbol{\tau}$ is stress tensor in both media;

$\boldsymbol{\mu}=-\boldsymbol{\mathcal{B}}\times\mathbf{S}$ is the couple tensor; $\boldsymbol{\mathcal{B}}$ is the tensor of exchange interactions;

$\rho\,\lambda\dot{\varphi}\mathbf{m}=\rho\,\mathbf{S}/\gamma$ is the kinetic moment, where λ is the density of axial moment of inertia, $\dot{\varphi}$ is the angular velocity of own rotation in Kelvin's medium; $\mathbf{S}$ is the magnetic moment, γ is the gyromagnetic ratio, $\rho\,S=M$ is the magnetization in ferromagnet.

[Note that λ needs to be infinitesimal and $\dot{\varphi}$ large for the analogy to work];

$\mathbf{L}=\mathbf{B}^e\times\mathbf{m}$ is the density of external body moment; $\mathbf{B}^e$ is the external magnetic induction;

$\boldsymbol{\tau}_\times=M\mathbf{B}^L\times\mathbf{m}$; $\mathbf{B}^L$ is the local magnetic induction in ferromagnet [only if $\mathcal{U}$ does not depend on α].

Exchange interaction corresponds in Kelvin's medium to a moment acting upon a particle depending on the relative turn of particles; $\boldsymbol{\tau}_{\times}$ (and $\mathbf{B}^{L}$ respectively), namely spin-lattice interaction, correspond to a moment depending only on the orientation of the particle under consideration (as if all other point bodies were point masses).

The coupling between angular and translational displacements in Kelvin's medium corresponds to the magnetoacoustic phenomena. Therefore it seems to be very important to properly take into account the dependence of $\mathcal{U}$ on the "mixed" kinds of deformations such as α (if we use (35)) or $\mathbf{F}$ (if we use (33)). These phenomena are most interesting both from theoretical and practical points of view.

3.2.2 Linear equations

To obtain linear equations it is more convenient to use (33) as opposed to (35) because tensor $\mathbf{\Phi} = o(\theta^2)$ when $\theta = o(1)$. G.A. Maugin [7] does not follow this way and his results are different from (56). If we put in (56) ${}^4\mathbf{Y} = 0$, we get equations linearized relatively to ferromagnetic phase obtained in [7]. We can see that our expression is more general because it includes the term coupling elastic, spin-lattice and exchange interactions. This coupling occurs in real magnetic solids, and sometimes it results in formation of helicoidal magnetic structures [11]. We can suppose that taking this term into account is significant for description of magnetoacoustic resonance.

In case of ferromagnets in (61) tensor ${}^4\mathbf{T}$ is the tensor of elastic constants, ${}^4\mathbf{U}$ depends on the magnetizability constants, ${}^4\mathbf{W}$ can be expressed through piezomagnetic constants, and ${}^4\mathbf{Z}$ can be expressed through ferromagnetic exchange constants.

Linear dynamic equations are the same as (62) but we have to put $\mu = 0$ in the right-hand side of the second equation since the analogy between dynamic equations of ferromagnets and Kelvin's medium exists if λ, μ are infinitesimal and $\dot{\varphi}$ is large. Setting $\mu = 0$, we lose the influence of the initial conditions.

Let us consider the case when the external magnetic induction $\mathbf{B}^e$ can be expressed as $B_0\mathbf{m}_0 + \tilde{\mathbf{B}}$, where $\tilde{\mathbf{B}}$ is infinitesimal and $\tilde{\mathbf{B}} \cdot \mathbf{m}_0 = 0$. Then external moment $\mathbf{L}$ can be written as

$$\mathbf{L} = S\mathbf{m} \times \mathbf{B}^e = S(\mathbf{m}_0 + \theta \times \mathbf{m}_0) \times (B_0\mathbf{m}_0 + \tilde{\mathbf{B}}) + O\,(\theta \cdot \tilde{\mathbf{B}}) = S(\mathbf{m}_0 \times \tilde{\mathbf{B}} - B_0\theta) + O\,(\theta \cdot \tilde{\mathbf{B}})$$

We obtain linear dynamic equations:

$$\begin{aligned}
&\overset{\circ}{\nabla} \cdot ({}^4\mathbf{X} \cdot\cdot (\overset{\circ}{\nabla}\mathbf{u} + \theta \times \mathbf{E}) + {}^4\mathbf{Y} \cdot\cdot \overset{\circ}{\nabla}\theta)^{\top} + \rho\,\mathbf{Q} = \rho\,\ddot{\mathbf{u}},\\
&\overset{\circ}{\nabla} \cdot ((\overset{\circ}{\nabla}\mathbf{u} + \theta \times \mathbf{E}) \cdot\cdot {}^4\mathbf{Y} + {}^4\mathbf{Z} \cdot\cdot \overset{\circ}{\nabla}\theta)^{\top} + \\
&\quad + ({}^4\mathbf{X} \cdot\cdot (\overset{\circ}{\nabla}\mathbf{u} + \theta \times \mathbf{E}) + {}^4\mathbf{Y} \cdot\cdot \overset{\circ}{\nabla}\theta)^{\top}_{\times} + M\mathbf{m}_0 \times \tilde{\mathbf{B}} - MB_0\theta = M\dot{\theta} \times \mathbf{m}_0.
\end{aligned} \qquad (74)$$

One can see that B_0 acts as a torsional spring.

4 Wave processes

We shall consider wave propagation in the medium with infinitesimal angles of nutation for the case when exact analogy between ferromagnets and Kelvin's medium can be established, i.e. λ, μ are infinitesimal, $\dot{\varphi}$ is large, $\lambda\dot{\varphi} = O(1)$. The results can be interpreted both in terms of Kelvin's medium and ferromagnets.

Let us consider the case $\mathbf{Q} = 0$, $\tilde{\mathbf{B}} = 0$. We shall search for a solution of (74) in the form $\mathbf{u} = \mathbf{u}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\Omega t)}$, $\boldsymbol{\theta} = \boldsymbol{\theta}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\Omega t)}$. After substituting into (62) we obtain a spectral problem:

$$\begin{aligned}
&({}^4\mathbf{X}_1 \cdot\cdot\, \mathbf{k}\otimes\mathbf{k} - \rho\,\Omega^2\,\mathbf{E})\cdot\mathbf{u}_0 + (i\,\mathbf{k}\cdot{}^4\mathbf{X}_3 \cdot\cdot\, \boldsymbol{\epsilon} + {}^4\mathbf{Y}_1 \cdot\cdot\, \mathbf{k}\otimes\mathbf{k})\cdot\boldsymbol{\theta}_0 = 0,\\
&({}^4\mathbf{Y}_2 \cdot\cdot\, \mathbf{k}\otimes\mathbf{k} - i\,\boldsymbol{\epsilon}\cdot\cdot\,{}^4\mathbf{X}\cdot\mathbf{k})\cdot\mathbf{u}_0 + \\
&+({}^4\mathbf{Z}_1 \cdot\cdot\, \mathbf{k}\otimes\mathbf{k} + \boldsymbol{\epsilon}\cdot\cdot\,{}^4\mathbf{X}\cdot\cdot\,\boldsymbol{\epsilon} + MB_0\mathbf{a} + i\,(2({}^3\tilde{\mathbf{N}}\cdot\mathbf{k})^A + M\mathbf{m}_0\times\mathbf{E}))\cdot\boldsymbol{\theta}_0 = 0,
\end{aligned} \tag{75}$$

where

$$ {}^4\mathbf{X}_1 = X^{mnkl}\,\mathbf{r}_m\otimes\mathbf{r}_k\otimes\mathbf{r}_l\otimes\mathbf{r}_n, \qquad {}^4\mathbf{X}_3 = X^{mnkl}\mathbf{r}_n\otimes\mathbf{r}_m\otimes\mathbf{r}_k\otimes\mathbf{r}_l, \tag{76}$$

$$ {}^4\mathbf{Y}_1 = Y^{mn\beta l}\mathbf{r}_m\otimes\mathbf{r}_\beta\otimes\mathbf{r}_l\otimes\mathbf{r}_n, \qquad {}^4\mathbf{Y}_2 = Y^{mnkl}\,\mathbf{r}_k\otimes\mathbf{r}_m\otimes\mathbf{r}_n\otimes\mathbf{r}_l, \tag{77}$$

$$ {}^3\tilde{\mathbf{N}} = -\boldsymbol{\epsilon}\cdot\cdot\,{}^4\mathbf{N}, \qquad {}^4\mathbf{Z}_1 = Z^{\alpha n\beta l}\,\mathbf{r}_\alpha\otimes\mathbf{r}_\beta\otimes\mathbf{r}_l\otimes\mathbf{r}_n. \tag{78}$$

Let us consider a material with a transversal isotropy (for highly symmetric media we have $\mathbf{Y} = 0$). Let $\mathbf{m}_0$ be an axis of isotropy. We shall investigate the particular case when $\mathbf{X}$ is isotropic, and $\mathbf{Z}$ is orthotropic. In this case taking into account restrictions (59) given by 6th balance equation in the linear theory (48) we have

$$ {}^4\mathbf{X} = (X^{11} - X^{22})\mathbf{E}\otimes\mathbf{E} + 2X^{22}(\mathbf{r}_m\otimes\mathbf{r}_n)^S(\mathbf{r}^m\otimes\mathbf{r}^n)^S \tag{79}$$

$$ \mathbf{H} = H^{11}\mathbf{a}\otimes\mathbf{a} + H^{22}(\mathbf{a}_2\otimes\mathbf{a}_2 + \mathbf{a}_4\otimes\mathbf{a}_4), \tag{80}$$

where $\mathbf{a}_2 = \mathbf{r}_1\otimes\mathbf{r}_1 - \mathbf{r}_2\otimes\mathbf{r}_2$ and $\mathbf{a}_4 = \mathbf{r}_1\otimes\mathbf{r}_2 + \mathbf{r}_2\otimes\mathbf{r}_1$

$$ \mathbf{N} = N((\mathbf{r}_2\otimes\mathbf{r}_3)^A\otimes\mathbf{r}_1\otimes\mathbf{r}_3 + (\mathbf{r}_3\otimes\mathbf{r}_1)^A\otimes\mathbf{r}_2\otimes\mathbf{r}_3), \tag{81}$$

$$\begin{aligned}
\mathbf{Z} =& Z^{11}\mathbf{a}\otimes\mathbf{a} + Z^{33}\mathbf{a}_3\otimes\mathbf{a}_3 + Z^{22}(\mathbf{a}_2\otimes\mathbf{a}_2 + \mathbf{a}_4\otimes\mathbf{a}_4) + \\
&+ Z^{1313}(\mathbf{r}_1\otimes\mathbf{r}_3\otimes\mathbf{r}_1\otimes\mathbf{r}_3 + \mathbf{r}_2\otimes\mathbf{r}_3\otimes\mathbf{r}_2\otimes\mathbf{r}_3)
\end{aligned} \tag{82}$$

where $\mathbf{a}_3 = \mathbf{r}_1\mathbf{r}_2 - \mathbf{r}_2\mathbf{r}_1$, and (75) can be rewritten as

$$\begin{pmatrix}
A_{11} - \rho\,\Omega^2 & A_{12} & A_{13} & B_{11} & B_{12}\\
A_{12} & A_{22} - \rho\,\Omega^2 & A_{23} & B_{21} & B_{22}\\
A_{13} & A_{23} & A_{33} - \rho\,\Omega^2 & B_{31} & B_{32}\\
B_{11} & B_{21} & B_{31} & C_{11} & C_{12} - iM\Omega\\
B_{12} & B_{22} & B_{32} & C_{21} + iM\Omega & C_{22}
\end{pmatrix}\cdot\begin{pmatrix} u_1\\ u_2\\ u_3\\ \theta_1\\ \theta_2\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\\0\end{pmatrix} \tag{83}$$

where $A_{mn}, B_{m\beta}, C_{\alpha\beta}$ are real polynomials of second degree in $\mathbf{k}$ (see Appendix A). A_{mn} depend only on $\mathbf{X}$ and $B_{m\beta}$ depend on $\mathbf{Y}$, $C_{\alpha\beta}$ depend on $\mathbf{Z}, M$, and B_0.

If $\mathbf{Y} = \mathbf{0}$, i.e. there is no coupling between angular and translational displacements (magnetic and elastic subsystems), possible dispersion relation graphs are shown in Fig. 2. There exists a cut-off frequency for angular (spin) waves. The position of the curve

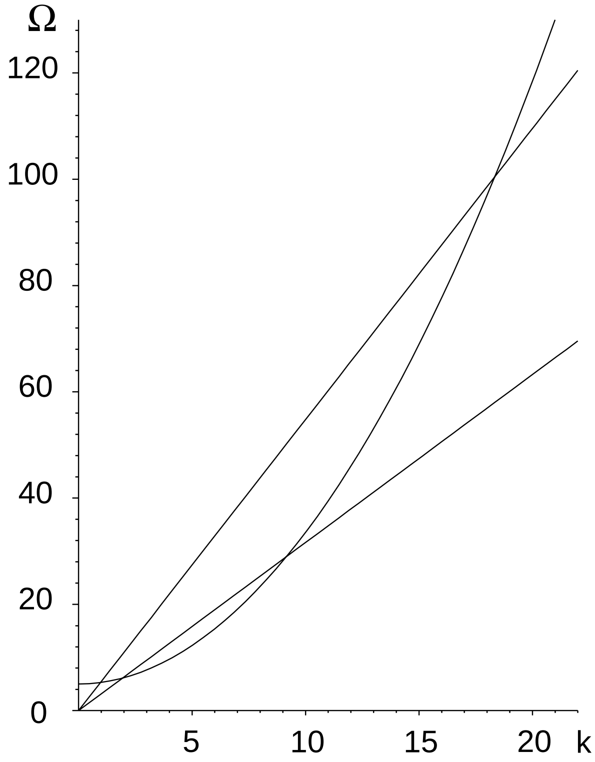

Figure 2: Partial dispersion curves

corresponding to these waves depends on magnetization M (or on the kinetic moment of a point body in Kelvin's medium); the curve may lay higher than other curves if M is not large enough. The cut-off frequency increases with increasing B_0. [Curves in Fig. 2 are calculated with parameters $M = 50,\ B_0 = 5,\ X^{11} = 20,\ X^{22} = 10,\ Z^{11} = 20,\ Z^{22} = 10,\ \mathbf{k} \cdot \mathbf{m}_0 = 0$].

Let us consider the case when $\mathbf{Y}$ is infinitesimal, i.e. the coupling between angular and translational displacements in Kelvin's medium (magnetic and elastic subsystems in ferromagnet) is weak.

The graphs for dispersion relations are close to the partial curves corresponding to the case of independent oscillations of elastic and magnetic subsystems. However, there is an essential difference: the coupling, even weak, qualitatively changes the behaviour of the curves in neighbourhoods of their intersections.

Indeed, suppose $B_{m\beta} \neq 0$ but infinitesimal. Let us consider equation (83) in possibly complex coordinates such that matrixes A and C are diagonal. After these changes of variables components of matrix B continue to be infinitesimal. Suppose the point (Ω^*, k^*) is an intersection of two graphs for partial dispersion equations. Consider a particular case when these roots of partial dispersion equations are not multiple. It means $f_1(\Omega^*, k^*) = f_2(\Omega^*, k^*) = 0$, where $f_1(\Omega, k)$ and $f_2(\Omega, k)$ are diagonal components of matrix after transformation, $k = |\mathbf{k}|$. In a neighbourhood of (Ω^*, k^*) problem (83) may

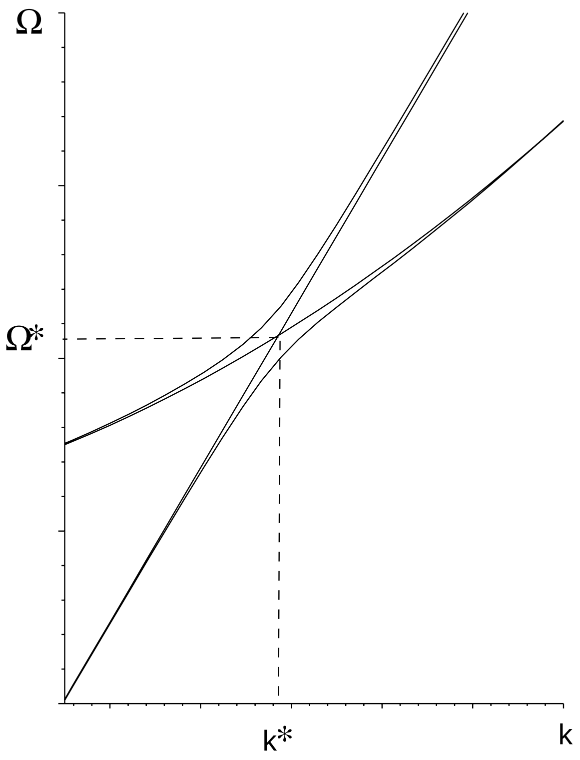

Figure 3: Dispersion curves for partial and coupled waves

be written as follows:

$$f_1(\Omega, k) f_2(\Omega, k) - b(\Omega, k) = 0, \tag{84}$$

where b is infinitesimal and depends on the other matrix components. Supposing that $b(\Omega^*, k^*) = b_* \neq 0$ (the coupling is not degenerative at this point) we shall expand f_1 and f_2. Let $\tilde{\Omega} = \Omega - \Omega^*$, $\tilde{k} = k - k^*$, $c_{g1} = -\frac{f'_{1k}}{f'_{1\Omega}}$, $c_{g2} = -\frac{f'_{2k}}{f'_{2\Omega}}$ (group velocities of non-coupled waves). Expanding to second order terms, (84) becomes

$$(\tilde{\Omega} - \tilde{k}(c_{g1} + c_{g2})/2)^2 - \tilde{k}^2 (c_{g1} - c_{g2})^2/4 = b_*. \tag{85}$$

It is a hyperbola with asymptotes coinciding with tangents to the partial curves $f_1 = 0$, $f_2 = 0$ at the point (Ω^*, k^*). We see that curves corresponding to the weak coupled waves have no intersection in the neighbourhood of (Ω^*, k^*); they are close to each other and their group velocities are equal at k^* and at Ω^* (Fig. 3).

This behaviour of the dispersion relation graphs points to the phenomenon called magnetoacoustic resonance in the physics of ferromagnets. Elastic and spin waves interact with each other in the neighbourhood of this point; for example, it is possible to excite elastic waves with an external magnetic field or to induce a spin wave with an elastic one. This phenomenon has a lot of applications in technology.

Since we take into account couplings of elastic and spin waves in the most general way, we see more principal possibilities for existence of phenomena akin to magnetoacoustic resonance. In the material under consideration there can be four points analogous to the point of magnetoacoustic resonance (the number of this points depends on the direction of $\mathbf{k}$). The intersection of compression wave graph and magnetic waves corresponds to the case described above. The intersection of shear wave graphs and magnetic wave graphs needs a separate analysis since partial shear curve corresponds to a double root in the case of isotropic $\mathbf{X}$.

Graphs in Fig. 4 are calculated with parameters $\mathsf{M} = 50, \mathsf{B}_0 = 5, \mathsf{X}^{11} = 20, \mathsf{X}^{22} = 10$, $\mathsf{Z}^{11} = 20, \mathsf{Z}^{22} = 10, \mathsf{H}^{11} = \mathsf{H}^{22} = 1$, $\mathbf{k} \cdot \mathbf{m}_0 = 0$, and in Fig. 2 with the same parameters but ${}^4\mathbf{H} = \mathbf{0}$. In the latter case all resonances disappear. One can see that the assumption of G.A. Maugin ${}^4\mathbf{Y} = \mathbf{0}$ may exclude from consideration some essential phenomena.

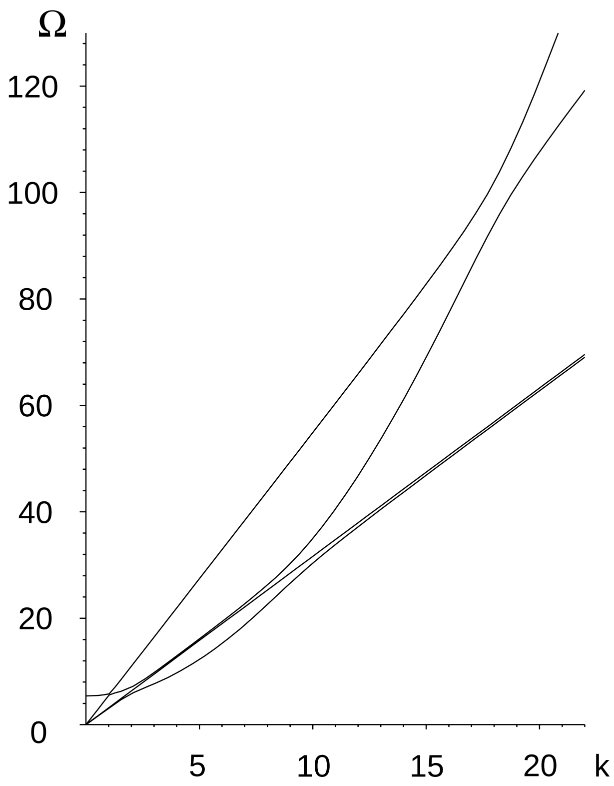

Figure 4: Dispersion curves: four resonances

To understand clearly these phenomena one has to consider nonlinear theory. The most difficult problem is to find the concrete form of nonlinear strain energy. We have seen that it is very important to take into account all kinds of deformation that can provide coupling between elastic and magnetic subsystem.

5 Conclusion

In this paper we obtain nonlinear constitutive and dynamic equations of Kelvin's medium. We show an exact analogy between elastic ferromagnetic insulators and this medium. We consider all kinds of deformations that can induce internal stresses, which gives the possibility to take into account the interaction between magnetic and elastic subsystems in the most general way. This is important for description of a magnetoacoustic resonance. For example, there can be found more resonances for some directions of wave propagation in low symmetric materials.

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Appendix A

$$
\begin{aligned}
A_{11} &= k_1^2(X^{11} + X^{22}) + (k_2^2 + k_3^2)X^{22} \\
A_{22} &= k_2^2(X^{11} + X^{22}) + (k_1^2 + k_3^2)X^{22} \\
A_{33} &= k_3^2(X^{11} + X^{22}) + (k_1^2 + k_2^2)X^{22} \\
A_{12} &= k_1 k_2 X^{11} \\
A_{13} &= k_1 k_3 X^{11} \\
A_{23} &= k_2 k_3 X^{11} \\
B_{11} &= k_1^2(H^{11} + H^{22}) + k_2^2 H^{22} \\
B_{22} &= k_2^2(H^{11} + H^{22}) + k_1^2 H^{22} \\
B_{12} = B_{21} &= k_1^2 k_2^2 H^{11} + k_3^2 N/2 \\
B_{31} &= -k_2 k_3 N/2 \\
B_{32} &= -k_1 k_3 N/2 \\
C_{11} &= k_1^2(Z^{11} + Z^{22}) + k_2^2(Z^{22} + Z^{33}) + k_3^2 Z^{1313} + MB_0 \\
C_{22} &= k_2^2(Z^{11} + Z^{22}) + k_1^2(Z^{22} + Z^{33}) + k_3^2 Z^{1313} + MB_0 \\
C_{12} &= k_1 k_2(Z^{11} - Z^{33})
\end{aligned}
\tag{86}
$$