A Micro-Polar Theory for Piezoelectric Materials*

Abstract

Theory of the piezoelectric materials had been developed many years ago. There exist a several theories of the piezoelectricity. All of them lead to the very complicated equations. The exact solutions of these equations may be found only for very particular cases. By this reason it is not easy to compare theoretical and experimental results. At the present time it seems to be possible to say that there is no qualitative discrepancies between theory and experiments. From the pure theoretical point of view in the theory of the piezoelectricity there are some serious problems. It was supposed that the stress state of the piezoelectric material can be described by means of the symmetrical stress tensor. However some piezoelectric materials are the dipole crystals. In such a case the rotation degrees of freedom must be taken into account. It means that the theory of the piezoelectric materials must be constructed on the base of the micro-polar continuum. The theory of such a kind is presented in the report. The basic equations are derived from the fundamental laws of Eulerian mechanics and contain two unsymmetrical stress tensors. The theory presented in the report differs from conventional theory very significantly. However under some assumptions this theory may be reduced to the classical one. The theory was tested on some simple problems and results were compared with classical ones.

1 Classical set of equations

Here we briefly present main equations of the classical theory.

Equation of motion:

$$\nabla \cdot \boldsymbol{\tau} + \rho \mathbf{F} = \rho \ddot{\mathbf{u}}, \qquad (1)$$

where $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathsf{T}}$ is the symmetric stress tensor, ρ is the mass density, $\mathbf{u}$ is the displacement vector.

Poisson equation:

$$\nabla \cdot \mathbf{D} = 0, \quad \nabla \cdot \mathbf{E}' = 0, \qquad (2)$$

where $\mathbf{D} = \mathbf{E}' + 4\pi\mathbf{P}$ is the vector of electric displacement density, $\mathbf{E}'$ is the vector of electric field intensity in vacuum, $\mathbf{P}$ is the vector of density of induced polarization.

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The energy balance equation in classics has the following form:

$$\rho\dot{\mathbb{U}} = \boldsymbol{\tau}\cdot\cdot\dot{\varepsilon} + \mathbf{E}\cdot\dot{\mathbf{D}}, \tag{3}$$

The electric enthalpy density is expressed by

$$\mathbb{F} = \mathbb{U} - \mathbf{E}\cdot\mathbf{D}. \tag{4}$$

For the linear approximation the following biquadratic form is employed:

$$\rho\mathbb{F} = \rho\mathbb{F}_0 + \frac{1}{2}\varepsilon\cdot\cdot\mathbf{C}\cdot\cdot\varepsilon - \mathbf{E}\cdot\mathbf{M}\cdot\cdot\varepsilon - \frac{1}{2}\mathbf{E}\cdot\boldsymbol{\epsilon}\cdot\mathbf{E}. \tag{5}$$

Constitutive equations are expressed by:

$$\boldsymbol{\tau} = \mathbf{C}\cdot\cdot\varepsilon - \mathbf{E}\cdot\mathbf{M}, \tag{6}$$

$$\mathbf{D} = \mathbf{M}\cdot\cdot\varepsilon + \boldsymbol{\epsilon}\cdot\mathbf{E}, \tag{7}$$

where $\mathbf{E}$ is the vector electric field intensity in medium, ε is the tensor of linear deformation, $\mathbf{C}$ is the tensor of elasticity, $\mathbf{M}$ is the tensor of piezoelectricity, $\boldsymbol{\epsilon}$ is the tensor of dielectric permittivity.

This notation uses displacements and electric field as independent variables. The other notation assumes, that independent variables are displacements and electric displacement. This notation of classical theory uses the intrinsic energy instead of the electric enthalpy. When equation

$$\rho\mathbb{U} = \rho\mathbb{U}_0 + \frac{1}{2}\varepsilon\cdot\cdot\mathbf{C}^{(c)}\cdot\cdot\varepsilon + \mathbf{D}\cdot\mathbf{M}^{(c)}\cdot\cdot\varepsilon + \frac{1}{2}\mathbf{D}\cdot\boldsymbol{\epsilon}^{(c)}\cdot\mathbf{D}, \tag{8}$$

is used, then the constitutive equations are expressed by equations:

$$\boldsymbol{\tau} = \mathbf{C}^{(c)}\cdot\cdot\varepsilon + \mathbf{D}\cdot\mathbf{M}^{(c)}, \tag{9}$$

$$\mathbf{E} = \mathbf{M}^{(c)}\cdot\cdot\varepsilon + \boldsymbol{\epsilon}^{(c)}\cdot\mathbf{D}. \tag{10}$$

2 Particle model

In the present work we present a micro-polar theory for mediums with non-zero electric dipole momentum density. Such a materials possesses piezoelectric properties, i.e. the electric field influence on the mechanical state of medium. In order to write equations for piezoelectric media, we have presented the particle model of such media. We use lagrangian description.

Let the particles be point bodies with dipole properties. The particles have abilities to move in space and rotate. Let the particle have the ability to change the value of dipole as well, i.e. the particle is elastic point-body. Let the particle with dipole value $\mathbf{d}_0$ in the reference configuration be characterized by following parameters: $\mathbf{R}_0^+$ and $\mathbf{R}_0^-$ are the vectors of charges q^+ and q^-, respectively ($q^+ = -q^- = q$ are the charge values); $\mathbf{r}_0$ is the radius of geometrical center of the particle.

Let in the actual configuration charges $q^\pm$ be moved from points $\mathbf{R}_0^\pm$ to points $\mathbf{R}^\pm$. Respectively, the center of particle is moved to the point $\mathbf{r}$. Let us introduce the following notations:

$$\mathbf{u} = \mathbf{r} - \mathbf{r}_0, \tag{11}$$

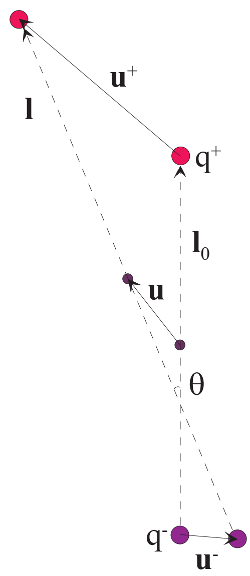

Figure 1: Point-body as a couple of charges

$$\mathbf{u}^{+} = \mathbf{R}^{+} - \mathbf{R}_0^{+}, \tag{12}$$

$$\mathbf{u}^{-} = \mathbf{R}^{-} - \mathbf{R}_0^{-}. \tag{13}$$

$$\mathbf{d}_0 = q\mathbf{l}_0 = q(\mathbf{R}_0^{+} - \mathbf{R}_0^{-}), \tag{14}$$

$$\mathbf{d} = q\mathbf{l} = q(\mathbf{R}^{+} - \mathbf{R}^{-}), \tag{15}$$

Let $\boldsymbol{\theta}(|\boldsymbol{\theta}| \ll 1)$ be the turn of the vector $\mathbf{d}_0$ to $\mathbf{d}$. Let $\mathbf{p} \equiv \mathbf{d} - \mathbf{d}_0$ be the change in dipole state. Let δ be the relative dipole value change

$$|\mathbf{d}| = |\mathbf{d}_0|(1+\delta). \tag{16}$$

After certain transformations for $\mathbf{p}$ we have decomposition

$$\mathbf{p} \simeq \mathbf{p}_1 + \mathbf{p}_2, \tag{17}$$

where

$$\mathbf{p}_1 = \delta\mathbf{d}_0, \qquad \mathbf{p}_2 = (1+\delta)\boldsymbol{\theta} \times \mathbf{d}_0. \tag{18}$$

Let us define the piezoelectric polarization density of continuum as

$$\mathcal{P}^p = \lim_{\Delta V \to 0} \frac{\sum_{k \in \Delta V}(\mathbf{p}_{1k} + \mathbf{p}_{2k})}{\Delta V} = \mathcal{P}_1^p + \mathcal{P}_2^p, \tag{19}$$

where

$$\mathcal{P}_1^p = \delta\mathcal{P}^s, \qquad \mathcal{P}_2^p = (1+\delta)\boldsymbol{\theta} \times \mathcal{P}^s, \tag{20}$$

where $\mathcal{P}^s$ is the density of spontaneous polarization.

The polarization vector $\mathcal{P}^p$ is the sum of two orthogonal polarization vectors of different nature: the first one is associated with rotation of medium particle, the second one is associated with changing of its absolute value.

Let us now write the power, given to the point-body by effective electric field. For this purpose we use the Lorentz formulae and Poisson equation. Finally, after transformations we have:

$$\dot{\mathbf{e}} = (\boldsymbol{\nabla} \cdot \mathbf{E})\mathbf{d}_0 \cdot \dot{\mathbf{u}} + \mathbf{E} \cdot \dot{\mathbf{p}}. \tag{21}$$

Using equations (18), let us write the time derivative of the vector (17):

$$\dot{\mathbf{p}} = \dot{\delta}(\mathbf{d}_0 + \theta \times \mathbf{d}_0) + (1+\delta)\dot{\theta} \times \mathbf{d}_0 \simeq \dot{\delta}\mathbf{d}_0 + \dot{\theta} \times \mathbf{d}_0. \tag{22}$$

Now, the equation for medium may be written:

$$\dot{\mathcal{E}} = (\boldsymbol{\nabla} \cdot \mathbf{E})\mathcal{P}^s \cdot \dot{\mathbf{u}} + (1+\delta)(\mathcal{P}^s \times \mathbf{E}) \cdot \dot{\theta} + (\mathcal{P}^s \cdot \mathbf{E})\dot{\delta}. \tag{23}$$

3 The Laws of motion

Kinetic energy for the point body with inertia tensor **J** is represented in the form:

$$\mathcal{K} = \frac{1}{2}m\dot{\mathbf{u}}^2 + \frac{1}{2}\boldsymbol{\omega} \cdot \mathbf{Q}(\mathbf{t}) \cdot \mathbf{J} \cdot \mathbf{Q}(\mathbf{t})^T \cdot \boldsymbol{\omega}. \tag{24}$$

The density of momentum:

$$\mathbf{K}_1 = \frac{\partial \mathcal{K}}{\partial \dot{\mathbf{u}}} = \rho\dot{\mathbf{u}},$$

where $\rho = V^{-1} \sum\limits_V m_i$ is the mass density, V is material volume. The equation of momentum balance is expressed by

$$\frac{d}{dt}\int_V \mathbf{K}_1 \, dV = \frac{d}{dt}\int_V \rho\dot{\mathbf{u}} dV = \int_V \rho\mathbb{F} dV + \int_S \mathbf{T}_{(n)} dS, \tag{25}$$

where $\mathbb{F}$ is external force density, $\mathbf{T}_{(n)}$ is stress vector. The following formulae is true:

$$\mathbf{T}_{(n)} = \mathbf{n} \cdot \mathbf{T}, \tag{26}$$

where **n** is normal vector, **T** is Cauchy stress tensor. Let us apply Green theorem

$$\int\limits_S \mathbf{T}_{(n)} d\mathbf{S} = \int\limits_S \mathbf{n} \cdot \mathbf{T} d\mathbf{S} = \int\limits_V \boldsymbol{\nabla} \cdot \mathbf{T} dV$$

and, taking into account (26), the equation (25) become

$$\int_V [\rho\ddot{\mathbf{u}} - \rho\mathbb{F} - \boldsymbol{\nabla} \cdot \mathbf{T}] \, dV = 0. \tag{27}$$

The momentum balance equation:

$$\boldsymbol{\nabla} \cdot \mathbf{T} + \rho\mathbb{F} = \rho\ddot{\mathbf{u}}. \tag{28}$$

The other form of this equation is expressed by

$$\boldsymbol{\nabla}\cdot\boldsymbol{\tau}-\frac{1}{2}\boldsymbol{\nabla}\times\mathbf{q}+\rho\mathbb{F}=\rho\ddot{\mathbf{u}}, \qquad (29)$$

where $\boldsymbol{\tau}=\frac{1}{2}(\mathbf{T}+\mathbf{T}^{\mathrm{T}})$, and $\mathbf{q}=\mathbf{T}_{\times}$.

For turn-tensor $\mathbf{Q}$ it is possible to write the following approximation:

$$\mathbf{Q}\approx\mathbf{I}+\boldsymbol{\phi}\times\mathbf{I}, \qquad \boldsymbol{\phi}=\phi\mathbf{e}_{\phi}. \qquad (30)$$

Using Poisson equation

$$\dot{\mathbf{Q}}=\boldsymbol{\omega}\times\mathbf{Q}, \qquad (31)$$

the expression for angular velocity follows:

$$\boldsymbol{\omega}=\dot{\boldsymbol{\phi}}.$$

For small $\boldsymbol{\omega}$, the equation for kinetic momentum is expressed by:

$$\mathbf{K}_2=\mathbf{r}\times\mathbf{K}_1+\frac{\partial\mathcal{K}}{\partial\dot{\boldsymbol{\phi}}}=\rho(\mathbf{r}\times\dot{\mathbf{u}}+\mathbf{J}\cdot\dot{\boldsymbol{\phi}}(x,t)). \qquad (32)$$

Integral form of the second Law of dynamics is as follows:

$$\frac{d}{dt}\int_V\mathbf{K}_2\,dV=\int_V\rho\left(\mathbf{r}\times\mathbb{F}+\mathbb{L}\right)dV+\int_S\left(\mathbf{r}\times\mathbf{T}_{(n)}+\boldsymbol{\mu}_{(n)}\right)dS, \qquad (33)$$

where $\mathbb{L}$ is the external momentum, $\boldsymbol{\mu}_{(n)}$ is the momentum stress tensor. The Cauchy equation for $\boldsymbol{\mu}$ is:

$$\boldsymbol{\mu}_{(n)}=\mathbf{n}\cdot\boldsymbol{\mu}. \qquad (34)$$

The local form of the second Law is expressed by:

$$\boldsymbol{\nabla}\cdot\boldsymbol{\mu}+\mathbf{q}+\rho\mathbb{L}=\rho\mathbf{J}\cdot\ddot{\boldsymbol{\phi}}. \qquad (35)$$

Let us suppose

$$\boldsymbol{\mu}=\mathbf{m}\times\mathbf{I}. \qquad (36)$$

This is not the only available representation, but it is not so important today to specify it more precisely. Finally we have the following equation:

$$\boldsymbol{\nabla}\times\mathbf{m}+\mathbf{q}+\rho\mathbb{L}=\rho\mathbf{J}\cdot\ddot{\boldsymbol{\phi}}. \qquad (37)$$

The integral form of the energy balance equation is expressed by:

$$\frac{d}{dt}\int_V\left(\frac{1}{2}\rho\dot{\mathbf{u}}^2+\frac{1}{2}\rho\dot{\boldsymbol{\phi}}\cdot\mathbf{J}\cdot\dot{\boldsymbol{\phi}}+\rho\mathbb{U}\right)dV=\int_V\left(\rho\mathbb{F}\cdot\dot{\mathbf{u}}+\rho\mathbb{L}\cdot\dot{\boldsymbol{\phi}}+Q\right)dV+$$
$$+\int_S\left(\mathbf{T}_{(n)}\cdot\dot{\mathbf{u}}+\boldsymbol{\mu}_{(n)}\cdot\dot{\boldsymbol{\phi}}+\mathbf{H}\cdot\mathbf{n}\right)dS, \qquad (38)$$

where $\mathbf{H}$ is energy flow vector, Q is density of external energy supply sources. Generally, Q represent the energy dissipation.

The local form of energy balance equation is represented as follows:

$$\rho\dot{\mathbb{U}}=\boldsymbol{\tau}\cdot\cdot\dot{\boldsymbol{\varepsilon}}-\mathbf{q}\cdot\dot{\boldsymbol{\theta}}-\mathbf{m}\cdot\dot{\boldsymbol{\gamma}}+\boldsymbol{\nabla}\cdot\mathbf{H}+Q. \qquad (39)$$

where

$$\boldsymbol{\varepsilon}\equiv\frac{1}{2}\left(\boldsymbol{\nabla}\mathbf{u}+\boldsymbol{\nabla}\mathbf{u}^{\mathrm{T}}\right), \quad \boldsymbol{\theta}\equiv\boldsymbol{\phi}-\frac{1}{2}\boldsymbol{\nabla}\times\mathbf{u}, \quad \boldsymbol{\gamma}=\boldsymbol{\nabla}\times\boldsymbol{\phi}. \qquad (40)$$

4 Equations of piezoelectric medium.

Let electric field be the only external influence. Then, from equation (23), we can obtain:

$$\rho\mathbb{F} = (\boldsymbol{\nabla}\cdot\mathbf{E})\boldsymbol{\mathcal{P}}^s, \tag{41}$$

$$\rho\mathbb{L} = (1+\delta)\boldsymbol{\mathcal{P}}^s \times \mathbf{E}. \tag{42}$$

The last term in equation (23) comes directly to the intrinsic energy equation.

$$Q = (\boldsymbol{\mathcal{P}}^s\cdot\mathbf{E})\dot{\delta}. \tag{43}$$

Let us introduce two scalars: the temperature T and the entropy S. Let that values satisfy the following equation:

$$T\dot{S} = \boldsymbol{\nabla}\cdot\mathbf{H} + Q_i, \tag{44}$$

where Q_i represent the work of dissipative forces

$$Q_i = \boldsymbol{\tau}_i(\boldsymbol{\varepsilon},\dot{\boldsymbol{\varepsilon}})\cdot\cdot\dot{\boldsymbol{\varepsilon}} - \mathbf{q}_i(\boldsymbol{\theta},\dot{\boldsymbol{\theta}})\cdot\dot{\boldsymbol{\theta}} - \mathbf{m}_i(\boldsymbol{\gamma},\dot{\boldsymbol{\gamma}})\cdot\dot{\boldsymbol{\gamma}}. \tag{45}$$

For heat flow vector **H** it is possible to write equation:

$$\mathbf{H} = -\chi\boldsymbol{\nabla}T. \tag{46}$$

The equation for intrinsic energy may be rewritten:

$$\rho\dot{\mathbb{U}} = \boldsymbol{\tau}\cdot\cdot\dot{\boldsymbol{\varepsilon}} - \mathbf{q}\cdot\dot{\boldsymbol{\theta}} - \mathbf{m}\cdot\dot{\boldsymbol{\gamma}} + (\mathbf{E}\cdot\boldsymbol{\mathcal{P}}^s)\dot{\delta} + T\dot{S}. \tag{47}$$

Let us assume the hypothesis of natural state and represent the intrinsic energy as positively defined bilinear quadratic form:

$$\begin{aligned}\rho\mathbb{U} = \rho\mathbb{U}_0 &+ \frac{1}{2}\boldsymbol{\varepsilon}\cdot\cdot\mathbf{C}^{(\varepsilon)}\cdot\cdot\boldsymbol{\varepsilon} + \frac{1}{2}\boldsymbol{\theta}\cdot\mathbf{C}^{(\theta)}\cdot\boldsymbol{\theta} + \frac{1}{2}\boldsymbol{\gamma}\cdot\mathbf{C}^{(\gamma)}\cdot\boldsymbol{\gamma} + \frac{1}{2}C^{(\delta)}\delta^2 + \frac{1}{2}C^{(S)}S^2 + \\ &+ \boldsymbol{\theta}\cdot\mathbf{C}^{(\theta\varepsilon)}\cdot\cdot\boldsymbol{\varepsilon} + \boldsymbol{\gamma}\cdot\mathbf{C}^{(\gamma\varepsilon)}\cdot\cdot\boldsymbol{\varepsilon} + \delta\mathbf{C}^{(\delta\varepsilon)}\cdot\cdot\boldsymbol{\varepsilon} + S\mathbf{C}^{(S\varepsilon)}\cdot\cdot\boldsymbol{\varepsilon} + \\ &+ \boldsymbol{\gamma}\cdot\mathbf{C}^{(\gamma\theta)}\cdot\boldsymbol{\theta} + \delta\mathbf{C}^{(\delta\theta)}\cdot\boldsymbol{\theta} + S\mathbf{C}^{(S\theta)}\cdot\boldsymbol{\theta} + \\ &+ \delta\mathbf{C}^{(\delta\gamma)}\cdot\boldsymbol{\gamma} + S\mathbf{C}^{(S\gamma)}\cdot\boldsymbol{\gamma} + C^{(\delta S)}\delta S.\end{aligned} \tag{48}$$

Then, it is possible to write Cauchy-Green relations:

$$\boldsymbol{\tau} = \frac{\partial\rho\mathbb{U}}{\partial\boldsymbol{\varepsilon}} = \mathbf{C}^{(\varepsilon)}\cdot\cdot\boldsymbol{\varepsilon} + \boldsymbol{\theta}\cdot\mathbf{C}^{(\theta\varepsilon)} + \boldsymbol{\gamma}\cdot\mathbf{C}^{(\gamma\varepsilon)} + \mathbf{C}^{(\delta\varepsilon)}\delta + \mathbf{C}^{(S\varepsilon)}S, \tag{49}$$

$$-\mathbf{q} = \frac{\partial\rho\mathbb{U}}{\partial\boldsymbol{\theta}} = \mathbf{C}^{(\theta\varepsilon)}\cdot\cdot\boldsymbol{\varepsilon} + \mathbf{C}^{(\theta)}\cdot\boldsymbol{\theta} + \boldsymbol{\gamma}\cdot\mathbf{C}^{(\gamma\theta)} + \mathbf{C}^{(\delta\theta)}\delta + \mathbf{C}^{(S\theta)}S, \tag{50}$$

$$-\mathbf{m} = \frac{\partial\rho\mathbb{U}}{\partial\boldsymbol{\gamma}} = \mathbf{C}^{(\gamma\varepsilon)}\cdot\cdot\boldsymbol{\varepsilon} + \mathbf{C}^{(\gamma\theta)}\cdot\boldsymbol{\theta} + \mathbf{C}^{(\gamma)}\cdot\boldsymbol{\gamma} + \mathbf{C}^{(\delta\gamma)}\delta + \mathbf{C}^{(S\gamma)}S, \tag{51}$$

$$\mathbf{E}\cdot\boldsymbol{\mathcal{P}}^s = \frac{\partial\rho\mathbb{U}}{\partial\delta} = \mathbf{C}^{(\delta\varepsilon)}\cdot\cdot\boldsymbol{\varepsilon} + \mathbf{C}^{(\delta\theta)}\cdot\boldsymbol{\theta} + \mathbf{C}^{(\delta\gamma)}\cdot\boldsymbol{\gamma} + C^{(\delta)}\delta + C^{(\delta S)}S, \tag{52}$$

$$T = \frac{\partial \rho \mathbb{U}}{\partial S} = \mathbf{C}^{(S\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + \mathbf{C}^{(S\theta)} \cdot \boldsymbol{\theta} + \mathbf{C}^{(S\gamma)} \cdot \boldsymbol{\gamma} + C^{(\delta S)}\delta + C^{(S)}S. \tag{53}$$

The equation of motion are as follows:

$$\nabla \cdot \boldsymbol{\tau} - \frac{1}{2}\nabla \times \mathbf{q} + (\nabla \cdot \mathbf{E})\boldsymbol{\mathcal{P}}^s = \rho\ddot{\mathbf{u}}, \tag{54}$$

$$\nabla \times \mathbf{m} + \mathbf{q} + (1+\delta)\boldsymbol{\mathcal{P}}^s \times \mathbf{E} = \rho\mathbf{J} \cdot \ddot{\boldsymbol{\phi}}, \tag{55}$$

$$-\chi\nabla^2 T + Q_i = T\dot{S}. \tag{56}$$

5 Micropolar theory: transformation to classical form

In order to compare the set of equations described above with equations of classical theory, let us try to make some assumptions and simplify the micropolar theory. Let us rewrite equation (19), taking into account (20), and suppose $\delta \ll 1$:

$$\boldsymbol{\mathcal{P}}^p = \delta\boldsymbol{\mathcal{P}}^s + \boldsymbol{\theta} \times \boldsymbol{\mathcal{P}}^s. \tag{57}$$

From this equation it is obvious, that for known δ and $\boldsymbol{\theta}$ it is possible to find $\boldsymbol{\mathcal{P}}^p$. Reverse task has no single solution. Let us suppose $\boldsymbol{\theta} \cdot \boldsymbol{\mathcal{P}}^s = 0$. Then, from (57) it is possible to write:

$$\boldsymbol{\theta} = \frac{\boldsymbol{\mathcal{P}}^s \times \boldsymbol{\mathcal{P}}^p}{|\boldsymbol{\mathcal{P}}^s|^2} = \boldsymbol{\chi}^s \times \boldsymbol{\mathcal{P}}^p, \tag{58}$$

$$\delta = \frac{\boldsymbol{\mathcal{P}}^s \cdot \boldsymbol{\mathcal{P}}^p}{|\boldsymbol{\mathcal{P}}^s|^2} = \boldsymbol{\chi}^s \cdot \boldsymbol{\mathcal{P}}^p. \tag{59}$$

where $\boldsymbol{\chi}^s = \boldsymbol{\mathcal{P}}^s/|\boldsymbol{\mathcal{P}}^s|^2$. Assume, that $\boldsymbol{\phi} = 0$ (this assumption is valid for crystalline structures) and neglect the temperature effects. Equation (55) rewrite as follows:

$$\mathbf{q} = -\boldsymbol{\mathcal{P}}^s \times \mathbf{E}. \tag{60}$$

Then, relations (49)–(52) become:

$$\boldsymbol{\tau} = \mathbf{C}^{(\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + \boldsymbol{\mathcal{P}}^p \cdot (-\boldsymbol{\chi}^s \times \mathbf{C}^{(\theta\varepsilon)} + \boldsymbol{\chi}^s \otimes \mathbf{C}^{(\delta\varepsilon)}), \tag{61}$$

$$\boldsymbol{\mathcal{P}}^s \times \mathbf{E} = \mathbf{C}^{(\theta\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + (\mathbf{C}^{(\theta)} \times \boldsymbol{\chi}^s + \mathbf{C}^{(\delta\theta)} \otimes \boldsymbol{\chi}^s) \cdot \boldsymbol{\mathcal{P}}^p, \tag{62}$$

$$\boldsymbol{\mathcal{P}}^s \cdot \mathbf{E} = \mathbf{C}^{(\delta\varepsilon)} \cdot\cdot \boldsymbol{\varepsilon} + (\mathbf{C}^{(\delta\theta)} \times \boldsymbol{\chi}^s + \mathbf{C}^{(\delta)} \otimes \boldsymbol{\chi}^s) \cdot \boldsymbol{\mathcal{P}}^p, \tag{63}$$

From equations (62) and (63) it follows the expression for $\mathbf{E}$:

$$\begin{aligned}\mathbf{E} = &(-\boldsymbol{\chi}^s \times \mathbf{C}^{(\theta\varepsilon)} + \boldsymbol{\chi}^s \otimes \mathbf{C}^{(\delta\varepsilon)}) \cdot\cdot \boldsymbol{\varepsilon} + \\ &+ (-\boldsymbol{\chi}^s \times \mathbf{C}^{(\theta)} \times \boldsymbol{\chi}^s + \boldsymbol{\chi}^s \otimes \mathbf{C}^{(\delta\theta)} \times \boldsymbol{\chi}^s - \boldsymbol{\chi}^s \times \mathbf{C}^{(\delta\theta)} \otimes \boldsymbol{\chi}^s + \mathbf{C}^{(\delta)}\boldsymbol{\chi}^s \otimes \boldsymbol{\chi}^s) \cdot \boldsymbol{\mathcal{P}}^p.\end{aligned}$$

Terms $\chi^s \otimes \mathbf{C}^{(\delta\theta)} \times \chi^s = 0$ and $\chi^s \times \mathbf{C}^{(\delta\theta)} \otimes \chi^s = 0$. This is obvious from the symmetry of the system: either the vectors χ^s and $\mathbf{C}^{(\delta\theta)}$ must be collinear at the given point of medium or vector $\mathbf{C}^{(\delta\theta)} = 0$ due to structure's symmetry.

Thus, we obtain the system of equations

$$\boldsymbol{\tau} = \mathbf{C}^{(n)} \cdot\cdot \varepsilon + \mathcal{P}^p \cdot \mathbf{M}^{(n)}, \tag{64}$$

$$\mathbf{E} = \mathbf{M}^{(n)} \cdot\cdot \varepsilon + \boldsymbol{\epsilon}^{(n)} \cdot \mathcal{P}^p. \tag{65}$$

The following notations are used:

$$\mathbf{C}^{(n)} = \mathbf{C}^{(\varepsilon)}, \tag{66}$$

$$\mathbf{M}^{(n)} = \chi^s \otimes \mathbf{C}^{(\delta\varepsilon)} - \chi^s \times \mathbf{C}^{(\theta\varepsilon)}, \tag{67}$$

$$\boldsymbol{\epsilon}^{(n)} = \mathbf{C}^{(\delta)} \chi^s \otimes \chi^s - \chi^s \times \mathbf{C}^{(\theta)} \times \chi^s. \tag{68}$$

The obtained equations (64)–(65) have similar shape compared to equations of classical theory (9)–(10). Moreover, comparison of the shape of mentioned material tensor with material tensors of classical theory shows the identical non-zero components placement, i.e. these material tensors are equivalent.

6 The system of equations for two-dimensional layer.

Let us consider the layer, made from material with two orthogonal planes of symmetry. Let thickness be along $\mathbf{e}_3$ direction. Then, for displacements we have equations:

$$\mathbf{u} = u_1(x_1, x_3)\mathbf{e}_1 + u_3(x_1, x_3)\mathbf{e}_3, \qquad \boldsymbol{\phi} = \phi_2(x_1, x_3)\mathbf{e}_2. \tag{69}$$

Let the spontaneous polarization and electric field be along the $\mathbf{e}_3$ axis:

$$\mathbf{P}^{(s)} = P^{(s)}\mathbf{e}_3, \qquad \mathbf{E} = E_3(x_1, x_3)\mathbf{e}_3. \tag{70}$$

The form of material tensors must be obtained, using the theory of symmetry. In order to simplify equations, here and further we will express δ from equation (52) and substitute it into equations (49), (50) and (51). Also, we will neglect temperature effects for the same reason. After transformations, the following system of 3 differential equations of the second order is obtained:

$$C_{11}\frac{\partial^2 u_1}{\partial x_1^2} + C_{12}\frac{\partial^2 u_1}{\partial x_3^2} + C_{13}\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + C_{14}\frac{\partial^2 \phi_2}{\partial x_1^2} + C_{15}\frac{\partial \phi_2}{\partial x_3} + C_{16}\frac{\partial^2 \phi_2}{\partial x_3^2} + \\ + C_{17}P^{(s)}\frac{\partial E_3}{\partial x_1} = \rho\frac{\partial^2 u_1}{\partial t^2}, \tag{71}$$

$$C_{21}\frac{\partial^2 u_3}{\partial x_1^2} + C_{22}\frac{\partial^2 u_3}{\partial x_3^2} + C_{23}\frac{\partial^2 u_1}{\partial x_1 \partial x_3} + C_{24}\frac{\partial \phi_2}{\partial x_1} + C_{25}\frac{\partial^2 \phi_2}{\partial x_1 \partial x_3} + C_{26}P^{(s)}\frac{\partial E_3}{\partial x_3} = \rho\frac{\partial^2 u_3}{\partial t^2}, \tag{72}$$

$$C_{31}\frac{\partial u_1}{\partial x_3} + C_{32}\frac{\partial u_3}{\partial x_1} + C_{33}\frac{\partial^2 u_1}{\partial x_1^2} + C_{34}\frac{\partial^2 u_1}{\partial x_3^2} + C_{35}\frac{\partial^2 u_3}{\partial x_1 \partial x_3} + C_{36}\frac{\partial^2 \phi_2}{\partial x_1^2} + C_{37}\frac{\partial^2 \phi_2}{\partial x_3^2} + \\ + C_{38}P^{(s)}\frac{\partial E_3}{\partial x_1} - C_2^{(\theta)}\phi_2 = \rho_2\frac{\partial^2 \phi_2}{\partial t^2}, \tag{73}$$

where the following notations are used:

$$C_{11} = C_{11}^{(\varepsilon)} - \frac{(C_1^{(\delta\varepsilon)})^2}{C^{(\delta)}}, \quad C_{12} = C_{55}^{(\varepsilon)} - C_{25}^{(\theta\varepsilon)} + \frac{1}{4}C_2^{(\theta)}, \quad C_{13} = C_{13}^{(\varepsilon)} + C_{55}^{(\varepsilon)} - \frac{C_1^{(\delta\varepsilon)} C_3^{(\delta\varepsilon)}}{C^{(\delta)}} - \frac{1}{4}C_2^{(\theta)},$$

$$C_{14} = C_{31}^{(\gamma\varepsilon)} - \frac{C_1^{(\delta\varepsilon)} C_3^{(\delta\gamma)}}{C^{(\delta)}}, \quad C_{15} = C_{25}^{(\theta\varepsilon)} - \frac{1}{2}C_2^{(\theta)}, \quad C_{16} = \frac{1}{2}C_6^{(\gamma\theta)} - C_{15}^{(\gamma\varepsilon)}, \quad C_{17} = \frac{C_1^{(\delta\varepsilon)}}{C^{(\delta)}},$$

$$C_{21} = C_{55}^{(\varepsilon)} + C_{25}^{(\theta\varepsilon)} + \frac{1}{4}C_2^{(\theta)}, \quad C_{22} = C_{33}^{(\varepsilon)} - \frac{(C_3^{(\delta\varepsilon)})^2}{C^{(\delta)}}, \quad C_{23} = C_{13}^{(\varepsilon)} + C_{55}^{(\varepsilon)} - \frac{C_1^{(\delta\varepsilon)} C_3^{(\delta\varepsilon)}}{C^{(\delta)}} - \frac{1}{4}C_2^{(\theta)},$$

$$C_{24} = C_{25}^{(\theta\varepsilon)} + \frac{1}{2}C_2^{(\theta)}, \quad C_{25} = C_{33}^{(\gamma\varepsilon)} - C_{15}^{(\gamma\varepsilon)} - \frac{1}{2}C_6^{(\gamma\theta)} - \frac{C_3^{(\delta\gamma)} C_3^{(\delta\varepsilon)}}{C^{(\delta)}}, \quad C_{26} = \frac{C_3^{(\delta\varepsilon)}}{C^{(\delta)}},$$

$$C_{31} = \frac{1}{2}C_2^{(\theta)} - C_{25}^{(\theta\varepsilon)}, \quad C_{32} = -\frac{1}{2}C_2^{(\theta)} - C_{25}^{(\theta\varepsilon)}, \quad C_{33} = C_{31}^{(\gamma\varepsilon)} - \frac{C_3^{(\delta\gamma)} C_1^{(\delta\varepsilon)}}{C^{(\delta)}},$$

$$C_{34} = \frac{1}{2}C_6^{(\gamma\theta)} - C_{15}^{(\gamma\varepsilon)}, \quad C_{35} = C_{33}^{(\gamma\varepsilon)} - C_{15}^{(\gamma\varepsilon)} - \frac{1}{2}C_6^{(\gamma\theta)} - \frac{C_3^{(\delta\gamma)} C_3^{(\delta\varepsilon)}}{C^{(\delta)}} = C_{25},$$

$$C_{36} = C_3^{(\gamma)} - \frac{(C_3^{(\delta\gamma)})^2}{C^{(\delta)}}, \quad C_{37} = C_1^{(\gamma)}, \quad C_{38} = \frac{C_3^{(\delta\gamma)}}{C^{(\delta)}}, \quad \rho_2 = \rho J_2.$$

Let us perform similar manipulation for the set of classical equations (1), (2), (6) and (7). After transformations the system looks as follows:

$$C_{11}^{(\varepsilon)} \frac{\partial^2 u_1}{\partial x_1^2} + C_{55}^{(\varepsilon)} \frac{\partial^2 u_1}{\partial x_3^2} + (C_{13}^{(\varepsilon)} + C_{55}^{(\varepsilon)}) \frac{\partial^2 u_3}{\partial x_1 \partial x_3} - M_{15} \frac{\partial E_1}{\partial x_3} - M_{31} \frac{\partial E_3}{\partial x_1} = \rho \frac{\partial^2 u_1}{\partial t^2}, \quad (74)$$

$$C_{55}^{(\varepsilon)} \frac{\partial^2 u_3}{\partial x_1^2} + C_{33}^{(\varepsilon)} \frac{\partial^2 u_3}{\partial x_3^2} + (C_{13}^{(\varepsilon)} + C_{55}^{(\varepsilon)}) \frac{\partial^2 u_1}{\partial x_1 \partial x_3} - M_{15} \frac{\partial E_1}{\partial x_1} - M_{33} \frac{\partial E_3}{\partial x_3} = \rho \frac{\partial^2 u_3}{\partial t^2}, \quad (75)$$

$$M_{15} \frac{\partial^2 u_3}{\partial x_1^2} + M_{33} \frac{\partial^2 u_3}{\partial x_3^2} + (M_{15} + M_{31}) \frac{\partial^2 u_1}{\partial x_1 \partial x_3} + \epsilon_1 \frac{\partial E_1}{\partial x_1} + \epsilon_3 \frac{\partial E_3}{\partial x_3} = 0. \quad (76)$$

Now let us consider another two-dimensional case. Let material be as in the previous case, with two orthogonal planes of symmetry. Let thickness be along $\mathbf{e}_3$ direction. Then, for displacements we have equations:

$$\mathbf{u} = u_1(x_1, x_3)\mathbf{e}_1 + u_3(x_1, x_3)\mathbf{e}_3, \qquad \boldsymbol{\phi} = \phi_2(x_1, x_3)\mathbf{e}_2. \quad (77)$$

Let the spontaneous polarization be along the $\mathbf{e}_1$ direction and electric field is parallel to the thickness direction:

$$\mathbf{P}^{(s)} = P^{(s)}\mathbf{e}_1, \qquad \mathbf{E} = E_3(x_1, x_3)\mathbf{e}_3.$$

Similarly to (71)–(73), the following system of equation is obtained:

$$C_{11} \frac{\partial^2 u_1}{\partial x_1^2} + C_{12} \frac{\partial^2 u_1}{\partial x_3^2} + C_{13} \frac{\partial^2 u_3}{\partial x_1 \partial x_3} + C_{14} \frac{\partial^2 \phi_2}{\partial x_1^2} + C_{15} \frac{\partial \phi_2}{\partial x_3} + C_{16} \frac{\partial^2 \phi_2}{\partial x_3^2} = 0, \quad (78)$$

$$C_{21} \frac{\partial^2 u_3}{\partial x_1^2} + C_{22} \frac{\partial^2 u_3}{\partial x_3^2} + C_{23} \frac{\partial^2 u_1}{\partial x_1 \partial x_3} + C_{24} \frac{\partial \phi_2}{\partial x_1} + C_{25} \frac{\partial^2 \phi_2}{\partial x_1 \partial x_3} = 0, \quad (79)$$

$$C_{31}\frac{\partial u_1}{\partial x_3}+C_{32}\frac{\partial u_3}{\partial x_1}+C_{33}\frac{\partial^2 u_1}{\partial x_1^2}+C_{34}\frac{\partial^2 u_1}{\partial x_3^2}+C_{35}\frac{\partial^2 u_3}{\partial x_1\partial x_3}+C_{36}\frac{\partial^2 \phi_2}{\partial x_1^2}+C_{37}\frac{\partial^2 \phi_2}{\partial x_3^2}+$$
$$+\left(C_1^{(\delta\varepsilon)}\frac{\partial u_1}{\partial x_1}+C_3^{(\delta\varepsilon)}\frac{\partial u_3}{\partial x_3}+C_3^{(\delta\gamma)}\frac{\partial \phi_2}{\partial x_1}\right)\frac{P^{(s)}}{C^{(\delta)}}E_3-P^{(s)}E_3-C_2^{(\theta)}\phi_2=0. \quad (80)$$

Equations (78)–(80) differs from equations (71)–(73) by the presence of terms, proportional to electric field in equation (80). This situation occurs due to the geometry of the case: spontaneous dipole momentum is perpendicular to electric field direction. The set of classical equations is written as follows:

$$C_{11}^{(\varepsilon)}\frac{\partial^2 u_1}{\partial x_1^2}+C_{55}^{(\varepsilon)}\frac{\partial^2 u_1}{\partial x_3^2}+(C_{13}^{(\varepsilon)}+C_{55}^{(\varepsilon)})\frac{\partial^2 u_3}{\partial x_1\partial x_3}-M_{31}\frac{\partial E_3}{\partial x_1}=\rho\frac{\partial^2 u_1}{\partial t^2}, \quad (81)$$

$$C_{55}^{(\varepsilon)}\frac{\partial^2 u_3}{\partial x_1^2}+C_{33}^{(\varepsilon)}\frac{\partial^2 u_3}{\partial x_3^2}+(C_{13}^{(\varepsilon)}+C_{55}^{(\varepsilon)})\frac{\partial^2 u_1}{\partial x_1\partial x_3}-M_{33}\frac{\partial E_3}{\partial x_3}=\rho\frac{\partial^2 u_3}{\partial t^2}, \quad (82)$$

$$M_{15}\frac{\partial^2 u_3}{\partial x_1^2}+M_{33}\frac{\partial^2 u_3}{\partial x_3^2}+(M_{15}+M_{31})\frac{\partial^2 u_1}{\partial x_1\partial x_3}+\epsilon_3\frac{\partial E_3}{\partial x_3}=0. \quad (83)$$

The comparison of two systems of equations (78)–(80) and (81)–(83) shows significant difference of two theories. In the classical theory electric field comes into equations only via its derivatives, while in the micro-polar theory we find linear terms.

The obtained equations are rather complicated. Meanwhile, it is possible to reduce number of independent variables by using assumption for Cosserat medium ($\boldsymbol{\phi}=0$). From this point of view we are going to consider the following, easy case.

7 The solution for one-dimensional static case.

Let us consider the infinite plate in electric field. Let the material, from which the plate is made of, have two mirror planes of symmetry, crossed by $\mathbf{e}_3$ axis. This symmetry group is named mm2 and it represent such materials as $LiGaO_2$ and Li_2GeO_3. One of the planes of the mirror symmetry is parallel to the plane of the plate. Let spontaneous polarization lay in the same direction and electric field is parallel to the thickness along $\mathbf{n}=\mathbf{e}_2$ axis:

$$\mathbf{P}^{(s)}=P^{(s)}\mathbf{e}_3, \qquad \mathbf{E}=E_2\mathbf{e}_2.$$

The only independent variable is x_2. From equations (54)–(55), after transformations, it follows:

$$\left(C_{22}^{(\varepsilon)}-\frac{(C_2^{(\delta\varepsilon)})^2}{C^{(\delta)}}\right)\frac{\partial^2 u_2}{\partial x_2^2}=0, \quad (84)$$

$$\left(4C_{44}^{(\varepsilon)}-4C_{14}^{(\theta\varepsilon)}+C_1^{(\theta)}\right)\frac{\partial^2 u_3}{\partial x_2^2}=0, \quad (85)$$

$$\left(\frac{1}{2}C_1^{(\theta)}-C_{14}^{(\theta\varepsilon)}\right)\frac{\partial u_3}{\partial x_2}+\left(\frac{C_2^{(\delta\varepsilon)}C_3^{(\delta\gamma)}}{C^{(\delta)}}-C_{32}^{(\gamma\varepsilon)}\right)\frac{\partial^2 u_2}{\partial x_2^2}=P^{(s)}E_2. \quad (86)$$

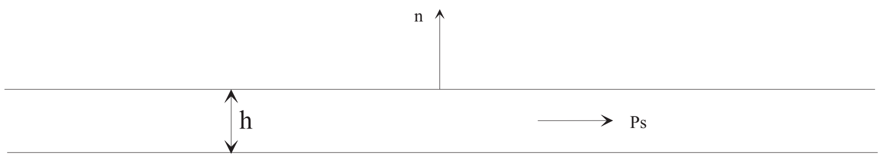

Figure 2: one-dimensional case

Choose the boundary conditions as follows:

$$x_2 = \pm h/2: \qquad \mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n} = 0, \tag{87}$$

and

$$x_2 = 0: \qquad u_2 = 0, \qquad u_3 = 0. \tag{88}$$

After transformations, (87) becomes:

$$x_2 = \pm h/2: \qquad \frac{\partial u_2}{\partial x_2} = 0. \tag{89}$$

Using (88), for u_2 we have:

$$u_2(x_2) = 0. \tag{90}$$

From (86) and (88) the solution for u_3 follows:

$$u_3(x_2) = P^{(s)}E_2x_2 \Big/ \left(\tfrac{1}{2}C_1^{(\theta)} - C_{14}^{(\theta\varepsilon)}\right). \tag{91}$$

7.1 The solution for one-dimensional classical case.

Let us consider equations (6)–(7). We will use here electric potential φ instead of electric field $\mathbf{E} = -\boldsymbol{\nabla}\varphi$. The system of equations in this case can be expressed by:

$$\frac{\partial^2 u_2}{\partial x_2^2} = 0, \qquad C_{44}\frac{\partial^2 u_3}{\partial x_2^2} + M_{24}\frac{\partial^2 \varphi}{\partial x_2^2} = 0, \qquad M_{24}\frac{\partial^2 u_3}{\partial x_2^2} - \epsilon_2\frac{\partial^2 \varphi}{\partial x_2^2} = 0. \tag{92}$$

The boundary conditions:

$$x_2 = \pm h/2: \qquad \mathbf{n}\cdot\boldsymbol{\tau}\cdot\mathbf{n} = 0, \qquad \mathbf{n}\cdot\mathbf{D} = \mathbf{n}\cdot\mathbf{D}_0,$$

$$x_2 = 0: \qquad u_2 = 0, \qquad u_3 = 0, \qquad \varphi = 0,$$

where $\mathbf{D}_0 = \epsilon_0\mathbf{E}$, ϵ_0 is the dielectric permittivity of vacuum. The boundary conditions are as follows:

$$\frac{\partial u_2}{\partial x_2} = 0, \qquad C_{44}\frac{\partial u_3}{\partial x_2} + M_{24}\frac{\partial \varphi}{\partial x_2} = 0, \qquad M_{24}\frac{\partial u_3}{\partial x_2} - \epsilon_2\frac{\partial \varphi}{\partial x_2} = E_2.$$

The solution of this system is as follows:

$$u_2 = 0, \qquad u_3 = \frac{M_{24}}{M_{24}^2 + C_{44}\epsilon_2}E_2x_2, \qquad \varphi = -\frac{C_{44}}{M_{24}^2 + C_{44}\epsilon_2}E_2x_2. \tag{93}$$

The solution for displacement $\mathbf{u}$ (90)–(91) and the solution (93) differs by constant multiplier. Both solutions contain material constants, but in the first case that constants are not yet known. Thus, the solutions seems to be equivalent.

8 Conclusion.

The micropolar theory of piezoelectricity has some important advantages compared to the classical one. Considered theory clearly shows the way how electric field influence on matter. There is possibility to consider inhomogeneous mediums by setting $\mathcal{P}^{s}(\mathbf{r})$ field. The micropolar theory allow to consider more general cases then classical one, adding new degrees of freedom. There is possibility to greatly simplify the micropolar theory by neglecting rotational degrees of freedom. Even after that simplification theory remains unsymmetrical and, thus, generally different compared to classic one. Meanwhile, unsymmetrical linear theory may lead to similar material tensor shapes and solutions, obtained by both theories.

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