Zhilin, P. A.; Sorokin, S. A.

The Motion of Gyrostat on the Nonlinear Elastic Foundation

The motion of the gyrostat with a tensor of inertia of a general form is considered. The rotation of rotor inside gyrostat is controlled by the motor of restricted power. The elastic foundation is specified by function of elastic energy. The transient and stationary regimes of the motion are studied. This report is based on the works [1–3].

1. The basic equations

The simplest kind of a gyrostat is a combination of rigid body, called stator in what follows, and an axially symmetric rigid body, called rotor and placed inside the stator. The rotation of the rotor does not change the distribution of mass of gyrostat. Let us accept that a stator is clamped in an elastic foundation. Let a tensor $\underline{\underline{P}}$ and a vector $\underline{\theta}$ be a tensor of turn and a vector of turn of the stator. They can be represented as

$$\underline{\underline{P}} = \underline{\underline{Q}}(\vartheta \underline{e}')\cdot\underline{\underline{Q}}(\beta\underline{k}), \qquad \underline{e}' = \underline{\underline{Q}}(\psi\underline{k})\cdot\underline{e}, \qquad \underline{e}\cdot\underline{k} = 0$$
$$\frac{2\sin\theta}{\theta}\,\underline{\theta} = \sin\vartheta(1+\cos\beta)\underline{e}' + \sin\beta(1+\cos\vartheta)\underline{k} + \sin\vartheta\sin\beta\,\underline{e}'\times\underline{k}, \qquad 1+2\cos\theta = \operatorname{tr}\underline{\underline{P}} \tag{1}$$

where unit vector $\underline{k}$ is a unit vector of axes of symmetry of the rotor when the elastic foundation is undeformed; $\theta = |\underline{\theta}|$; $\underline{\underline{Q}}(\vartheta\underline{e}')$ is a tensor of inclination of the stator. Let the vector $\underline{\omega}$ be the vector of angular velocity of the stator.

The tensor of turn of the rotor $\underline{\underline{P}}_R$ can be represented in the form

$$\underline{\underline{P}}_R = \underline{\underline{Q}}(\vartheta\underline{e}')\cdot\underline{\underline{Q}}(\tilde{\alpha}\underline{k}) = \underline{\underline{P}}(\underline{\theta})\cdot\underline{\underline{Q}}(\alpha\underline{k}), \qquad \alpha = \tilde{\alpha} - \beta \tag{2}$$

The tensor of inclination of a rotor coincides with one of the stator, α is the angle of turn of the rotor with respect to stator. The angular velocity $\underline{\omega}_R$ of a rotor can be found from the equality

$$\underline{\omega}_R = \underline{\omega} + \dot{\alpha}\underline{k}', \qquad \underline{k}' = \underline{\underline{P}}(\underline{\theta})\cdot\underline{k} = \underline{\underline{Q}}(\vartheta\underline{e}')\cdot\underline{k} \tag{3}$$

Let the tensor $\underline{\underline{A}}$ and $\underline{\underline{B}}$ be the tensors of inertia of stator and rotor in a reference position when the elastic foundation is underformed. It is supposed that gyrostat has a fixed point. The tensors $\underline{\underline{A}}$ and $\underline{\underline{B}}$ are defined with respect to this fixed point. For the tensor $\underline{\underline{B}}$ we have

$$\underline{\underline{B}} = B\underline{k}\otimes\underline{k} + B_1(\underline{\underline{E}} - \underline{k}\otimes\underline{k}) \quad\Rightarrow\quad \underline{\underline{P}}_R\cdot\underline{\underline{B}}\cdot\underline{\underline{P}}_R^T = \underline{\underline{P}}\cdot\underline{\underline{B}}\cdot\underline{\underline{P}}^T \tag{4}$$

The kinetic moment of the gyrostat is defined by the expression

$$\underline{L} = \underline{\underline{P}}\cdot\underline{\underline{A}}\cdot\underline{\underline{P}}^T\cdot\underline{\omega} + \underline{\underline{P}}_R\cdot\underline{\underline{B}}\cdot\underline{\underline{P}}_R^T\cdot\underline{\omega}_R = \underline{\underline{P}}\cdot\underline{\underline{J}}\cdot\underline{\underline{P}}^T\cdot\underline{\omega} + B\dot{\alpha}\underline{k}', \qquad \underline{\underline{J}} = \underline{\underline{A}} + \underline{\underline{B}} \tag{5}$$

For the elastic moment we have the formula [3]

$$\underline{M} = -\underline{\underline{Z}}^{-T}(\underline{\theta})\cdot\frac{dU}{d\underline{\theta}}, \qquad \underline{\underline{Z}}(\underline{\theta}) = \underline{\underline{E}} + \frac{1-\cos\theta}{\theta^2}\underline{\underline{R}} + \frac{\theta - \sin\theta}{\theta^3}\underline{\underline{R}}^2, \qquad \underline{\underline{R}} = \underline{\theta}\times\underline{\underline{E}} \tag{6}$$

For the transversally isotropic foundation we have

$$U(\underline{\theta}) = \mathcal{U}(\theta^2, \underline{k}\cdot\underline{\theta}), \qquad \underline{M} = -C_1\,\underline{\theta} - C_2\,\underline{\underline{Z}}^{-T}\cdot\underline{k} \tag{7}$$

$$C_1 = 2\,\frac{\partial\mathcal{U}}{\partial(\theta^2)}, \qquad C_2 = \frac{\partial\mathcal{U}}{\partial(\underline{k}\cdot\underline{\theta})} \tag{8}$$

Now we are able to write down the equations of motion

$$\underline{\underline{J}}\cdot\dot{\underline{\Omega}} + \underline{\Omega}\times\underline{\underline{J}}\cdot\underline{\Omega} + B\ddot{\alpha}\underline{k} + B\dot{\alpha}\underline{\Omega}\times\underline{k} + C_1\,\underline{\theta} + C_2\,\underline{\underline{Z}}^{-1}\cdot\underline{k} = \underline{0} \tag{9}$$

$$B(\ddot{\alpha} + \underset{\sim}{k}\cdot\dot{\underset{\sim}{\Omega}}) + \eta(\dot{\alpha} - \omega_m) = 0 \tag{10}$$

$$\dot{\underset{\sim}{\theta}} = \underset{\sim}{\Omega} + \frac{1}{2}\,\underset{\sim}{\theta}\times\underset{\sim}{\Omega} + \frac{1-g}{\theta^2}\,\underset{\sim}{\theta}\times(\underset{\sim}{\theta}\times\underset{\sim}{\Omega})\,, \qquad g = \frac{\theta\sin\theta}{2(1-\cos\theta)} \tag{11}$$

where $\underset{\sim}{\Omega} = \underset{\approx}{P}\cdot\underset{\sim}{\omega}$ is the right angular velocity of the stator, the equation (10) is a projection of the equation of motion of the rotor on the axis $\underset{\sim}{k}'$; $M_m = -\eta(\dot{\alpha} - \omega_m)$ is moment of motor, $\eta > 0$; ω_m is a nominal angular velocity of the motor. The system (9)–(11) gives a general description of motion of gyrostat on the elastic foundation.

2. The torsional vibrations of gyrostat

Let us consider the case when

$$\underset{\sim}{\theta} = \beta\underset{\sim}{k}\,, \qquad \underset{\approx}{J}\cdot\underset{\sim}{k} = J\underset{\sim}{k}\,, \qquad \underset{\sim}{\omega} = \underset{\sim}{\Omega} = \dot{\beta}\underset{\sim}{k}\,, \qquad \underset{\sim}{M}_e = -c(\beta)\beta\underset{\sim}{k}\,, \qquad 2\mathcal{U} = C_1\theta^2 + (C_2 - C_1)(\underset{\sim}{k}\cdot\theta)^2 \tag{12}$$

The system (9)–(10) takes the form

$$(J-B)\ddot{\beta} + c(\beta)\beta = \eta(\dot{\alpha} - \omega_m)\,, \qquad B\big(\dot{\alpha} - \omega_m\big)^{\cdot} + \eta(\dot{\alpha} - \omega_m) = -B\ddot{\beta} \tag{13}$$

The stationary rotations are defined by expressions $\alpha = \omega_m$, $\beta = 0$.

3. The small turns of stator

Let us consider the case when $|\underset{\sim}{\theta}| \ll 1$ and $\underset{\sim}{\theta} = \beta\underset{\sim}{k} + \underset{\sim}{y}(t)$, $\underset{\sim}{y}\cdot\underset{\sim}{k} = 0$. Let us accept for simplicity that the tensor of inertia $\underset{\approx}{A}$ is a transversally isotropic one: $\underset{\approx}{A} = A\underset{\sim}{k}\otimes\underset{\sim}{k} + A_1(\underset{\approx}{E} - \underset{\sim}{k}\otimes\underset{\sim}{k})$. In such a case in order to investigate the transient regime we have to solve the system

$$J_1\,\ddot{\underset{\sim}{y}} - B\dot{\alpha}\,\underset{\sim}{k}\times\dot{\underset{\sim}{y}} - B\ddot{\alpha}\,\underset{\sim}{k}\times\underset{\sim}{y} + C_1\,\underset{\sim}{y} = \underset{\sim}{0}\,, \qquad J_1 = A_1 + B_1 \tag{14}$$

$$J\ddot{\beta} + C_2\,\beta = -B\ddot{\alpha}\,, \qquad B(\ddot{\alpha} + \ddot{\beta}) + \eta(\dot{\alpha} - \omega_m) = 0\,, \qquad J = A + B \tag{15}$$

From (15) it follows:

$$t \to \infty \quad \Rightarrow \quad \beta \to 0, \quad \dot{\alpha} \to \omega_m \tag{16}$$

Thus for the large times we have the equation

$$J_1\,\ddot{\underset{\sim}{y}} - B\omega_m\,\underset{\sim}{k}\times\dot{\underset{\sim}{y}} + C_1\,\underset{\sim}{y} = \underset{\sim}{0} \tag{17}$$

This equation has a very simple solution — see [2].

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4. References

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Address: Prof. Dr. P. A. Zhilin. St. Petersburg State Technical University, Department of Theoretical Mechanics, Politechnicheskaya 29, St. Petersburg, 195251, Russia.