The Main Direction of the Development of Mechanics for XXI Century*

Abstract

The multi-spin continuum mechanics is an extension of the Cosserat continuum (single-spin continuum). The report presents a general theory and its applications to the derivation of the equations by Maxwell.

1 Introduction

Mechanics before Newton had been remaining by a collection of many important, but separated facts. Newton was the first, who set up a problem of construction of mechanics, as a science of the first principles. As the first principles Newton pointed out the three Laws of Motion, but he did not consider them as a sufficient foundation for a general construction of mechanics. For example, in work [1] Newton wrote: "Vis inertia is a passive principle, by means of which bodies stay in their motion or rest, receive motion, proportional applied to them force, and resist so, as far as meet a resistance (this is the statement of all three laws, P. Zh.). Only because of this there could not be a motion in the world. Other principle was necessary to reduce bodies in motion and, since they are in motion, one more principle is required for preservation of motion. For from various additions of two motions it is quite clear, that in the world there is not always the same momentum. If two balls, joint thin rod, rotate round a common center of gravity by uniform movement, while the center is uniformly gone on a direct line conducted in a plane of their circular movement, the sum of motions of two balls in that case, when the balls are on a direct line circumscribed by their center of gravity, will be more, than the sum of their motions, when they are on a line, perpendicular to this direct. From this example it is clear, that the motion can be received and to be lost" — see p.301. These words were written in 1717 and give clear impression about a level of development of mechanics in the first quarter of the XVIII century. The programmed Newton's idea about construction of mechanics on the base of the first principles had played a huge stimulating role. Euler carried out the realization of this program. In period with 1732 on 1755 Euler has developed the concept, which now is accepted as

*Zhilin P.A. The Main Direction of the Development of Mechanics for XXI Century // Lecture prepared for presentation at XXVIII Summer School – Conference, St. Petersburg, Russia, 2000.

Newtonian mechanics. In this concept the translation of mechanics on language of the differential equations was made. The stage of the construction of Newtonian mechanics in the fundamental plan was finished by the memoir of Euler[2] "Discovery of a new principle of mechanics", published in 1752. In that time Euler was considering, that the principle, opened by him, is possible to consider "as a unique base of mechanics and other sciences, which treat about movement of any bodies" — see [3]. Unfortunately, this point of view dominated in science down to 1925, when it finally had failed, and Newtonian mechanics was deprived with the status of fundamental science. Certainly, rather continuous period had proceeded to this end, when Newtonian mechanics was not able to describe the important physical concepts. Probably, for the first time this problem has arisen in the investigations of J. Maxwell, when he tried to describe true, i.e. not induced, magnetism, but it was not possible. Finally this period was finished by the creation of quantum mechanics.

Between that, in 1771, L. Euler not only clearly had realized an incompleteness of Newtonian mechanics, but also had indicated the path of its extension. In Newtonian mechanics there is only one form of motion, namely translation motion described by transposition of a body-point in space. However in many natural processes spinor motions play the main role. In such a motion the body-point does not change the position in space, but has own rotation. The spinor motions are the main method of accumulation and preservation of energy in the Nature. Not surprised therefore, that Newtonian mechanics has appeared powerless at the level of the microcosmos, where the spinor motions in essence cannot be ignored. In 1776 Euler publishes memoir "New method of determination of motion of rigid bodies" [4], where two independent Laws of Dynamics are stated for the first time: the equation of balance of momentum and equation of balance of kinetic moment (or moment of momentum in accepted, but unsuccessful, terms). This work opens new era in mechanics. Under an appropriate development of ideas of this work modern physics would look completely differently. Unfortunately, the comprehension of ideas of Euler has taken place only at the last quarter of XX century. At the end of XVIII century only J. Lagrange had realized significance of Euler's work, but he had not agreed with its main conclusions. In essence problem was reduced to a possibility or impossibility of the proof of the Archimedes law of the lever. Lagrange, as opposed to Euler, considered that the law of the lever is a corollary of the Newton laws. A large part of extensive introduction to the treatise "Analytical mechanics" [5] Lagrange devotes to the proof of the law of the lever. The Lagrange proof looks rather convincingly, but contains an error, which was not trivial for that time. Namely, Lagrange as the principle of the sufficient basis used reasons of a symmetry, which, as it is well known now, are quite capable to replace by itself conservation laws. In particular, the symmetry concerning a turn round some axes is equivalent to the absence of the moment round the same axes. The Lagrange method of description of mechanics has made a great impression on scientific community. The stable, but faulty, point of view had established that Lagrange's mechanics is quite able to replace by itself Newtonian mechanics. Actually mechanics of Lagrange is a rather poor subclass of Newtonian mechanics and it can not be considered as self-sufficient science about natural phenomena. It follows, for example, from the fact that the fundamental concepts like space, time, forces, moments, energy and etc., are not discussed and can not be introduced into consideration in Lagrange's mechanics, where all these concepts are used, but are not determined. Unfortunately, many theorists with a mathematical kind of thinking obviously underestimate importance and complexity of originating here problems. Besides mechanics of Lagrange is not suitable for the description of open systems, what all-real systems are. All said is quite valid with respect to mechanics of Hamilton that has mathematical dignities, but is very poor from a physical point of view. Main defect of Lagrange-Hamilton mechanics is the false impression, created by these theories, about a completeness of classical mechanics in the fundamental plan and, therefore, about its boundedness. Just this false impression has allowed to M. Plank to say the following words [6]: "Today we must recognize that... frameworks of classical dynamics ... have appeared too narrow to envelop all those physical phenomena that do not lend to direct observation by our rough organs of sense... The proof of this conclusion is given to us by the crying contradiction, that come to light in the universal laws of heat radiation, between the classical theory and experience". This point of view had become conventional in physics. The mechanics had evaded from a discussion of these hard questions and continued researches on the important applied problems.

Let's remind one more statement of M. Plank[7]: "The mechanical phenomena, or movements of material points, and all set of the electrical and magnetic phenomena as a single unit are completely separated. This by two area settle (exhausted) all physics, as all other parts of physics — acoustics, optics, and heat — can be quite reduced on the mechanics and electrodynamics. Final association of these two last classes of the phenomena, that would present by itself the crown of a building of theoretical physics, still it is necessary to give to the future". This statement by M. Plank causes some objections. First of all, the mechanical phenomena are not reduced at all to movements of material points, i.e. to Newtonian mechanics. Leonard Euler proved the basic incompleteness of Newtonian mechanics in 1776. Further, satisfactory theory of the electromagnetic phenomena is not developed till now. At last, more than doubtfully, that association of these theories (even if they would exist) would be a completion of theoretical physics, for obviously there are phenomena leaving the frameworks of these theories in their modern kind. Nevertheless, problem of association of the mechanics and electrodynamics, specified by M. Plank, exists and should be solved. The situation existing in a mechanics and physics can be called paradoxical. On the one hand, there are actual phenomena, which can not be circumscribed within the framework of classical mechanics from the point of view of the first principles. On the other hand, nobody has shown an inaccuracy of these principles. From this it follows, that the principles of Newtonian mechanics are necessary, but not sufficient, for the full description of the known experimental facts. This means, that Newtonian mechanics should be extended by adding of new principles. The statement of these new principles should emanate from intuitive understanding of a nature of those phenomena, which can not be circumscribed by methods of Newtonian mechanics. Certainly, this very complex problem can not be solved by simple means and requires special researches. If the mechanics does not realize necessity of the indicated researches and will limit by the analysis traditional (let even very important) problems, then its future has not any perspectives. If someone doubts of this, he should pay attention to the prompt vanishing of mechanics in the educational and research programs at the end of XX century.

The present article grows out desires of the author to understand the electrical and magnetic phenomena from the point of view of the principles of mechanics. The analysis of the known facts has shown, that the spinor motions,which are absent in Newtonian mechanics, are necessary for a description of the electromagnetic phenomena. Because of this the author thought that the full description of electromagnetism could be executed in the frameworks of Eulerian mechanics. The brief exposition of the main principles of Eulerian mechanics can be found in the paper [8], where the spinor motion is entered in terms of a tensor of turn. The main properties and various representations of the tensor of turn are stated in [9, 10]. The description of classical electrodynamics is given in terms of mechanics in the paper [11]. Besides in [11] the explanation of the known fact about inapplicability of classical electrodynamics for the description of a structure of atom is given. Consequent investigations have shown necessity of an introduction of multi-spin particles. In other words, Eulerian mechanics requires some additions to describe a presupposed structure of atom. Said above, probably, explains a title of the given work. Nevertheless, when describing the main results, the author considers only the pure mechanical aspects, since electromagnetic interpretations of these results are far from a desirable definiteness up to now.

In order to avoid a misunderstanding let us consider the basic terms. In what follows all considerations take place with respect to an inertial system of reference[13].

Newton's Mechanics contains the laws of dynamics of spinless particles. The state of the particle is defined by the vector of position $\mathbf{R}(t)$, the vector of momentum $m\mathbf{R}(t)$, the total energy $U = K + const$, where $K = 0.5m\dot{\mathbf{R}}\cdot\dot{\mathbf{R}}$ is the kinetic energy. The change of the momentum is determined by the vector of force $\mathbf{F}$. Besides, there are derived quantities: the vector $\mathbf{R}\times m\dot{\mathbf{R}}$ is called the moment of momentum of the particle about the origin, the vector $\mathbf{R}\times\mathbf{F}$ is called the moment of the force $\mathbf{F}$. In Newton's Mechanics only so called central forces are admissible. The basic model of Newton's Mechanics is the harmonic oscillator. The basic equation of the simplest form is

$$m\ddot{\mathbf{R}}+c\mathbf{R} = \mathbf{0}. \tag{1}$$

There is no need to speak about other aspects of Newton's Mechanics.

Euler's Mechanics contains the laws of dynamics of single-spin particles. The motion of the single-spin particle is defined by the vector of position $\mathbf{R}(t)$ and by the tensor of turn $\mathbf{P}(t)$. The velocities can be found from the equations

$$\mathbf{V}(t) = \dot{\mathbf{R}}(t), \qquad \dot{\mathbf{P}}(t) = \boldsymbol{\omega}(t)\times\mathbf{P}(t), \tag{2}$$

where the second equation is called the Poisson equation [8]. The total energy U of the particle is the sum $U = K + const$, where the kinetic energy is determined by the quadratic form

$$K = \frac{1}{2}m\mathbf{V}\cdot\mathbf{V} + \mathbf{V}\cdot\mathbf{P}\cdot\mathbf{B}\cdot\mathbf{P}^{T}\cdot\boldsymbol{\omega} + \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^{T}\cdot\boldsymbol{\omega}, \tag{3}$$

where tensors of second rank $\mathbf{B}$ and $\mathbf{C} = \mathbf{C}^{T}$ are the inertia tensors in the reference position, the scalar m is the mass. Now we are able to introduce the vector of momentum $\mathbf{K}_1$ and the vector of kinetic moment $\mathbf{K}_2$ by means of the expressions

$$\mathbf{K}_1 = \partial K/\partial\mathbf{V} = m\mathbf{V} + \mathbf{P}\cdot\mathbf{B}\cdot\mathbf{P}^{T}\cdot\boldsymbol{\omega}, \tag{4}$$

$$\mathbf{K}_2 = \underline{\mathbf{R}\times\mathbf{K}_1} + \partial K/\partial\,\boldsymbol{\omega} = \underline{\mathbf{R}\times\mathbf{K}_1} + \mathbf{V}\cdot\mathbf{P}\cdot\mathbf{B}\cdot\mathbf{P}^{T} + \mathbf{P}\cdot\mathbf{C}\cdot\mathbf{P}^{T}\cdot\boldsymbol{\omega}, \tag{5}$$

where the underlined term is called the moment of momentum. In Euler's Mechanics the change of momentum is determined by the force **F** and the change of kinetic moment is determined by the vector of moment **M**

$$\mathbf{M} = \mathbf{R} \times \mathbf{F} + \mathbf{L}, \tag{6}$$

where the vector **L** is called the torque. In general case the torque can't be determined in terms of **F**. The first and the second laws of dynamics [4] in Euler's Mechanics have the form

$$\dot{\mathbf{K}}_1 = \mathbf{F}, \qquad \dot{\mathbf{K}}_2 = \mathbf{M} = \mathbf{R} \times \mathbf{F} + \mathbf{L}. \tag{7}$$

The more detailed definitions may be found in the paper[8]. The basic model in Euler's Mechanics is the model of rigid body oscillator. In the simplest case the equations of motion of rigid body oscillator, i.e. the rigid body on an elastic foundation, can be derived from the equations (4) − (7) under some assumptions about the elastic foundation [14]. These equations have the form [14]

$$A\,\dot{\boldsymbol{\omega}} + c\boldsymbol{\theta} = 0, \quad \dot{\boldsymbol{\theta}} = \boldsymbol{\omega} - \frac{1}{2}\,\boldsymbol{\theta} \times \boldsymbol{\omega} + \frac{1-g}{\theta^2}\,\boldsymbol{\theta} \times (\boldsymbol{\theta} \times \boldsymbol{\omega}), \quad g = \frac{\theta \sin\theta}{2\,(1-\cos\theta)}, \tag{8}$$

where $\boldsymbol{\theta}$ is a vector of turn [9, 10, 14] $\theta = |\boldsymbol{\theta}|$. We see that even in the simplest case equation (8) has much more complex form then equation (1) for a usual oscillator. However for the plane vibrations we have $\boldsymbol{\theta} \times \boldsymbol{\omega} = 0$. In such a case equation (8) coincides with equation (1). Note that equation (8) corresponds to rotational degrees of freedom only, i.e. the body has a fixed point. In general case we have some combination of equations like (1) and (8).

When speaking about Euler's Mechanics it is necessary to point out the contribution of C. Truesdell [15, 16] who had studied Euler's works published after 1766 and had made them the property of scientific community.

Mechanics of multi-spin particles will be considered in the next sections of the paper.

2 Kinematics and Dynamical Structures of the Multi-Spin Particle

The multi-spin particle A is the complex object consisting of a carrier body A_1 and rotors $A_i \quad (i = 2, 3, ..., N)$ inside of A_1. Let $\mathbf{R}_i \quad (i = 1, 2, ..., N)$ be the position vectors of the mass center of A_i and m_i is the mass of the particle A_i. Let's accept that the set of points $\mathbf{R}_i$ is a rigid body. Let $\mathbf{P}_i$ be the turn-tensors of the bodies A_i. Then we have

$$\mathbf{R}_i = \mathbf{R} + \mathbf{P}_1 \cdot \boldsymbol{\rho}_i, \quad \mathbf{R} = \frac{1}{m}\sum_{i=1}^{N} m_i \mathbf{R}_i, \quad m = \sum_{i=1}^{N} m_i, \tag{9}$$

where **R** is the center of mass of A, the vectors $\boldsymbol{\rho}_i$ are the position vectors of the mass center of A_i with respect to the point **R** in the reference position. Thus the motion of a multi-spin particle A is determined in terms of $3(N+1)$ scalar functions

$$\mathbf{R}(t), \quad \mathbf{P}_1(t), \quad \mathbf{P}_2(t), \quad ... \quad \mathbf{P}_N(t). \tag{10}$$

The velocities of the multi-spin particle are determined from the next equations

$$\mathbf{V}(t) = \dot{\mathbf{R}}(t), \quad \dot{\mathbf{P}}_i(t) = \omega_i(t) \times \mathbf{P}_i(t). \tag{11}$$

We shall consider that rotor A_i is the body of rotation with the axis of symmetry $\mathbf{n}'_i$, which is supposed to be fixed with respect to the carrier body A_1. Because of this we have

$$\mathbf{n}'_i = \mathbf{P}_1 \cdot \mathbf{n}_i, \quad i = 2, ..., N, \tag{12}$$

where $\mathbf{n}_i$ are determined in the reference position. The turn-tensor of the carrier body can be represented in many different, but equivalent, forms

$$\begin{aligned} &\mathbf{P}_1 = \mathbf{T}_2 \cdot \mathbf{Q}(\psi_2 \mathbf{n}_2) = \mathbf{T}_3 \cdot \mathbf{Q}(\psi_3 \mathbf{n}_3) = ... = \mathbf{T}_N \cdot \mathbf{Q}(\psi_N \mathbf{n}_N) \Rightarrow \\ &\mathbf{T}_i = \mathbf{P}_1 \cdot \mathbf{Q}^T(\psi_i \mathbf{n}_i), \end{aligned} \tag{13}$$

where $\mathbf{Q}(\psi_i \mathbf{n}_i)$ is the turn around axis $\mathbf{n}_i$ by the angle ψ_i, $\mathbf{T}_i$ is the turn around axis orthogonal to the axis $\mathbf{n}_i$. For the turn-tensors $\mathbf{P}_i$ we have the analogous expressions

$$\mathbf{P}_i = \mathbf{S}_i \cdot \mathbf{Q}(\varphi_i \mathbf{n}_i). \tag{14}$$

Since the axes $\mathbf{n}_i$ are fixed with respect to the carrier body A_1 we have the conditions

$$\mathbf{T}_i = \mathbf{S}_i \Rightarrow \mathbf{S}_i = \mathbf{P}_1 \cdot \mathbf{Q}^T(\psi_i \mathbf{n}_i).$$

Now the equations (14) take a form

$$\begin{aligned} &\mathbf{P}_i = \mathbf{P}_1 \cdot \mathbf{Q}^T(\psi_i \mathbf{n}_i) \cdot \mathbf{Q}(\varphi_i \mathbf{n}_i) = \mathbf{P}_1 \cdot \mathbf{Q}(\beta_i \mathbf{n}_i), \\ &\beta_i = \varphi_i - \psi_i, \quad i = 2, 3, ..., N, \end{aligned} \tag{15}$$

where β_i is the angle of the turn of the rotor A_1 with respect to the carrier body A_1. Thus we see that the motion of the multi-spin particle can be described in terms of $6 + N - 1$ scalar function, i.e. it has $N + 5$ degrees of freedom. In what follows we shall accept

$$\mathbf{P} \triangleq \mathbf{P}_1, \quad \boldsymbol{\omega} \triangleq \boldsymbol{\omega}_1. \tag{16}$$

Making use of (15) one can find

$$\boldsymbol{\omega}_i = \boldsymbol{\omega} + \mathbf{P} \cdot \dot{\beta}_i \mathbf{n}_i = \boldsymbol{\omega} + \dot{\beta}_i \mathbf{n}'_i, \quad \mathbf{n}'_i = \mathbf{P} \cdot \mathbf{n}_i, \quad i = 2, 3, ..., N. \tag{17}$$

Let's define the kinetic energy K_i of the body A_i by the quadratic form

$$K_i = \frac{1}{2} m_i \dot{\mathbf{R}}_i \cdot \dot{\mathbf{R}}_i + \frac{1}{2} \boldsymbol{\omega}_i \cdot \mathbf{P}_i \cdot \mathbf{C}_i \cdot \mathbf{P}_i^T \cdot \boldsymbol{\omega}_i, \tag{18}$$

where $\mathbf{C}_i$ is the central tensor of inertia of the body A_i in the reference positions. For the rotors we have

$$\mathbf{C}_i = \lambda_i \mathbf{n}_i \otimes \mathbf{n}_i + \mu_i (\mathbf{E} - \mathbf{n}_i \otimes \mathbf{n}_i), \quad i = 2, 3, ..., N, \tag{19}$$

where λ_i, μ_i are the axial central moment of inertia and the equatorial central moment of inertia of the rotor A_i respectively. From the equations (15) and (19) it follows

$$\mathbf{P}_i \cdot \mathbf{C}_i \cdot \mathbf{P}_i^T = \mathbf{P} \cdot \mathbf{C}_i \cdot \mathbf{P}^T. \tag{20}$$

From the equation (9) it follows

$$\mathbf{V}_i = \dot{\mathbf{R}_i} = \mathbf{V} + \omega \times (\mathbf{R}_i - \mathbf{R}). \tag{21}$$

The momentum $\mathbf{K}_{1i}$ of the body A_i is defined as

$$\mathbf{K}_{1i} = \frac{\partial K_i}{\partial \mathbf{V}_i} = m_i \mathbf{V}_i = m_i (\mathbf{V} + \boldsymbol{\omega} \times (\mathbf{R}_i - \mathbf{R})) = m_i \mathbf{V} + \mathbf{B}_i \cdot \boldsymbol{\omega},$$
$$\mathbf{B}_i = m_i (\mathbf{R} - \mathbf{R}_i) \times \mathbf{E}. \tag{22}$$

The momentum $\mathbf{K}_1$ of the multi-spin particle is defined by the expression

$$\mathbf{K}_1 = \sum_{i=1}^{N} \mathbf{K}_{1i} = m\mathbf{V} + \left(\sum_{i=1}^{N} \mathbf{B}_i \right) \cdot \boldsymbol{\omega} = m\mathbf{V}, \quad \sum_{i=1}^{N} \mathbf{B}_i = \mathbf{0}. \tag{23}$$

The second equality in (23) follows from (9). Let's calculate the kinetic moment $\mathbf{K}_{2i}$ of A_i

$$\mathbf{K}_{2i} = \mathbf{R}_i \times \mathbf{K}_{1i} + \frac{\partial K_i}{\partial \boldsymbol{\omega}_i}, \tag{24}$$

where the first term in the right side is called the moment of momentum and the second term will be called the dynamical spin or the own moment of momentum of A_i. Making use of the formulae (22), (18), (9), (17), (19), (20) one can obtain

$$\mathbf{K}_{2i} = m_i \mathbf{R}_i \times \mathbf{V} + \left(\mathbf{R}_i \times \mathbf{B}_i + \mathbf{P} \cdot \mathbf{C}_i \cdot \mathbf{P}^T \right) \cdot \boldsymbol{\omega} + \lambda_i \dot{\beta}_i \mathbf{n}_i', \quad \beta_1 = 0. \tag{25}$$

The kinetic moment of the multi-spin particle $\mathbf{K}_2$ is defined by the expression

$$\mathbf{K}_2 = \sum_{i=1}^{N} \mathbf{K}_{2i} = \mathbf{R} \times m\mathbf{V} + \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^T \cdot \boldsymbol{\omega} + \sum_{i=2}^{N} \lambda_i \dot{\beta}_i \mathbf{n}_i', \tag{26}$$

where the tensor $\mathbf{C}$ has a form

$$\mathbf{C} = \sum_{i=1}^{N} \left(\mathbf{C}_i - m_i (\mathbf{r} - \mathbf{r}_i) \times \mathbf{E} \times (\mathbf{r} - \mathbf{r}_i) \right), \tag{27}$$

and the vectors $\mathbf{r}$ and $\mathbf{r}_i$ determine the mass centers of the particle A and of the bodies A_i in the reference position respectively. The total kinetic energy of the multi-spin particle is determined by the expression

$$K = \frac{1}{2} m\mathbf{V} \cdot \mathbf{V} + \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^T \cdot \boldsymbol{\omega} + \frac{1}{2} \sum_{i=2}^{N} \lambda_i \left(\dot{\beta}_i^2 + 2 \dot{\beta}_i \boldsymbol{\omega} \cdot \mathbf{n}_i' \right). \tag{28}$$

Now we are able to write down the laws of motion of the multi-spin particle.

3 The Laws of Motion of a Multi-Spin Particle

The multi-spin particle has $N+5$ degrees of freedom. Thus we need to formulate $N+5$ equations to find the next unknown functions

$$\mathbf{R}(t), \quad \mathbf{P}(t), \quad \beta_i(t), \quad i = 2, 3, ..., N. \tag{29}$$

First of all, we must formulate the laws of dynamics by Euler.

The equation of the momentum balance or the first law of dynamics by Euler

$$\dot{\mathbf{K}}_1 = \mathbf{F}, \tag{30}$$

where $\mathbf{K}_1$ is determined by the expression (23), and the vector $\mathbf{F}$ is the force acting on the multi-spin particle.

The equation of the kinetic moment balance or the second law of dynamics by Euler

$$\dot{\mathbf{K}}_2 = \mathbf{R} \times \mathbf{F} + \mathbf{L}, \tag{31}$$

where the vector $\mathbf{L}$ is called the torque and in general case it can't be defined in terms of a force. The equations (30) and (31) give to us 6 equations. Thus we need to formulate $N-1$ additional equations. For this end let us consider

The equations of motions of the rotors A_i

$$\dot{\mathbf{K}}_{1i} = \mathbf{F}_i, \quad \dot{\mathbf{K}}_{2i} = \mathbf{R}_i \times \mathbf{F}_i + \mathbf{L}_i, \quad i = 2, 3, ..., N \tag{32}$$

where $\mathbf{F}_i$ and $\mathbf{L}_i$ are the force and the torque acting on the rotor A_i from the carrier body A_1. Let's represent the torque $\mathbf{L}_i$ in the next form

$$\mathbf{L}_i = L_{mi}\mathbf{n}'_i + \mathbf{L}^*_i, \quad \mathbf{n}'_i \cdot \mathbf{L}^*_i = 0, \quad L_{mi} = -\eta_i \left(\dot{\beta}_i - \omega_i\right), \quad \eta_i > 0, \tag{33}$$

where $\omega_i = \mathrm{const}$ and $\eta_i = \mathrm{const}$ are the parameters of the particle. Making use of the results of the previous section one can obtain

$$\dot{\mathbf{K}}_{2i} = \mathbf{R}_i \times \mathbf{F}_i + \left((\lambda_i - \mu_i)(\boldsymbol{\omega} \cdot \mathbf{n}'_i)\,\mathbf{n}'_i + \mu_i \boldsymbol{\omega} + \lambda_i \dot{\beta}_i \mathbf{n}'_i\right)^{\cdot}. \tag{34}$$

Substituting this expression into the second equation (32) and multiplying the resulting equation by the vector $\mathbf{n}'_i$ we obtain the additional $N-1$ equations

$$\lambda_i \left(\dot{\beta}_i + \boldsymbol{\omega} \cdot \mathbf{n}'_i\right)^{\cdot} + \eta_i \left(\dot{\beta}_i - \omega_i\right) = 0, \quad i = 2, 3, ..., N. \tag{35}$$

The equations (30), (31), (35) give to us the complete system of equations of motion for the multi-spin particle.

4 The Equation of the Energy Balance

Let us formulate the third fundamental law, i.e. the equation of the energy balance

$$(K + U_p)^{\cdot} = \mathbf{F} \cdot \mathbf{V} + \mathbf{L} \cdot \boldsymbol{\omega} + \delta, \tag{36}$$

where U_p is an intrinsic energy of the particle, δ is the velocity of the energy input into the particle. In what follows we shall consider that $U_p = \mathrm{const}$. This means that the particle does not contain the elastic elements. In the considered case it is easy to calculate δ. Indeed, multiplying (30) by the vector V and so on we obtain

$$\dot{K} = \mathbf{F} \cdot \mathbf{V} + \mathbf{L} \cdot \boldsymbol{\omega} - \sum_{i=2}^{N} \eta_i \dot{\beta}_i \left(\dot{\beta}_i - \omega_i \right). \tag{37}$$

From the comparison of the equations (37) and (36) we see

$$\delta = - \sum_{i=2}^{N} \eta_i \dot{\beta}_i \left(\dot{\beta}_i - \omega_i \right). \tag{38}$$

The quantity δ is generated by the external supply of energy, for example, by the electrical device.

5 Continuum of the Multi-Spin Particles. The law of the Particle Conservation

Let's consider some inertial system of reference. Let Z be a set of the multi-spin particle. Let V be some domain that is fixed with respect to the system of reference. The boundary of V is the closed surface $S = \partial V$. Let $\rho(\mathbf{x},t)$ be a number of the particles in the infinitely small neighborhood of the point $\mathbf{x} \subset V$ at the actual instant of time t

$$\rho(\mathbf{x},t) \geq 0. \tag{39}$$

Let's formulate the law of the conservation of the particles

$$\frac{d}{dt} \int_{(V)} \rho(\mathbf{x},t)\, dV = - \int_{(S)} \rho \mathbf{n} \cdot \mathbf{V} dS, \qquad \int_{(S)} \mathbf{n} \cdot (\rho \mathbf{V})\, dS = \int_{(V)} \nabla \cdot (\rho \mathbf{V})\, dV. \tag{40}$$

From this it follows

$$\frac{d\rho}{dt} + \nabla \cdot (\rho \mathbf{V}) = 0. \tag{41}$$

This is a local form of the law of the particles conservation.

6 The First Law of Dynamics by Euler

The domain V contain the next quantity of the momentum

$$\mathbf{K}_1^* = \int_{(V)} \rho(\mathbf{x},t)\, \mathbf{K}_1(\mathbf{x},t)\, dV(\mathbf{x}), \tag{42}$$

where the momentum of a particle $\mathbf{K}_1$ is defined by the expression (23). Then the first law of dynamics can be written in the form

$$\frac{d}{dt}\int\limits_{(V)} \rho \mathbf{K}_1 dV = \int\limits_{(V)} \rho \mathbf{F} dV + \int\limits_{(S)} \mathbf{T}_{(n)} dS - \int\limits_{(S)} \rho\,(\mathbf{n}\cdot\mathbf{V})\,\mathbf{K}_1 dV.$$

For the last term we have

$$\int\limits_{(S)} \mathbf{n}\cdot(\rho\mathbf{V}\otimes\mathbf{K}_1)\,dS = \int\limits_{(V)} \nabla\cdot(\rho\mathbf{V}\otimes\mathbf{K}_1)\,dV. \qquad (43)$$

Now the first law can be rewritten in such a form

$$\underbrace{\frac{d}{dt}\int\limits_{(V)} \rho \mathbf{K}_1 dV}_{O(\varepsilon^3)} = \underbrace{\frac{d}{dt}\int\limits_{(V)} [\rho\mathbf{F} - \nabla\cdot(\rho\mathbf{V}\otimes\mathbf{K}_1)]\,dV}_{O(\varepsilon^3)} + \underbrace{\int\limits_{(S)} \mathbf{T}_{(n)} dS}_{O(\varepsilon^2)}.$$

From this we see that the next equation is valid

$$\int\limits_{(S)} \mathbf{T}_{(n)} dS = \mathbf{0}. \qquad (44)$$

Making use of standard considerations the stress tensor can be introduced

$$\mathbf{T}_{(n)} = \mathbf{n}\cdot\mathbf{T}. \qquad (45)$$

Thus we have

$$\int\limits_{(V)} \left[(\rho\mathbf{K}_1)^{\cdot} - \rho\mathbf{F} + \nabla\cdot(\rho\mathbf{V})\,\mathbf{K}_1 + \rho\mathbf{V}\cdot\nabla\mathbf{K}_1 - \nabla\cdot\mathbf{T}\right] dV = \mathbf{0}.$$

In the local form the first law can be represented as

$$\nabla\cdot\mathbf{T} + \rho\mathbf{F} = \rho(\dot{\mathbf{K}}_1 + \mathbf{V}\cdot\nabla\mathbf{K}_1), \qquad \mathbf{K}_1 = m\mathbf{V}(\mathbf{x},t), \qquad (46)$$

where $m = \mathrm{const}$ is the mass of the particle that is placed in the point $\mathbf{x}$ at the actual instant of time. The quantity ρm is the mass density. In the right-hand side of the first equation (46) the material derivative of $\mathbf{K}_1$ is written.

7 The Second Law of Dynamics by Euler

The equation of the balance of the kinetic moment in the integral form can be written as

$$\frac{d}{dt}\int\limits_{(V)} \rho \mathbf{K}_2 dV = \int\limits_{(V)} \rho\,(\mathbf{R}\times\mathbf{F} + \mathbf{L})\,dV + \int\limits_{(S)} \left(\mathbf{R}\times\mathbf{T}_n + \mathbf{M}_{(n)}\right) dS - \int\limits_{(S)} \rho\,(\mathbf{n}\cdot\mathbf{V})\,\mathbf{K}_2 dS. \qquad (47)$$

where $\mathbf{K}_2$ is defined by the expression (26), $\mathbf{L}$ is a mass density of the external torque. By means of the standard consideration it is easy to derive the Cauchy formula

$$\mathbf{M}_{(n)} = \mathbf{n} \cdot \mathbf{M}. \tag{48}$$

and the local form of the second law

$$\nabla \cdot \mathbf{M} + \mathbf{T}_{\times} + \rho \mathbf{L} = \rho(\dot{\mathbf{K}}_2 + \mathbf{V} \cdot \nabla \mathbf{K}_2), \tag{49}$$

where $\mathbf{K}_2$ is the dynamical spin of a particle

$$\mathbf{K}_2 = \mathbf{P}(\mathbf{x},t)\cdot\mathbf{C} \cdot \mathbf{P}^T(\mathbf{x},t) \cdot \boldsymbol{\omega}(\mathbf{x},t) + \sum_{i=2}^{N} \lambda_i \dot{\beta}_i(\mathbf{x},t)\, \mathbf{n}_i'(\mathbf{x},t), \tag{50}$$

and the tensor $\mathbf{C}$ is defined by the expression (27). To this equation it is necessary to add the equation of motion for the rotors (35)

$$\lambda_i \left(\dot{\beta}_i(\mathbf{x},t) + \omega(\mathbf{x},t) \cdot \mathbf{n}_i'(\mathbf{x},t) \right)^{\cdot} + \eta_i \left(\dot{\beta}_i(\mathbf{x},t) - \omega_i(\mathbf{x}) \right) = 0, \quad i = 2, 3, ..., N. \tag{51}$$

8 The Equation of the Energy Balance

Let's introduce the total energy in the domain V

$$E = \int_{(V)} \rho(\mathcal{K} + \mathcal{U}) dV, \tag{52}$$

where $\mathcal{K}, \mathcal{U}$ are the density of the kinetic and intrinsic energy respectively.

The equation of the energy balance is the next statement

$$\frac{d}{dt} \int_{(V)} \rho(\mathcal{K} + \mathcal{U}) dV = \int_{(V)} \rho\left[\mathbf{F} \cdot \mathbf{V} + \mathbf{L} \cdot \boldsymbol{\omega} + q\right] dV + \int_{(S)} (\mathbf{T}_n \cdot \mathbf{V} + \mathbf{M}_n \cdot \boldsymbol{\omega} + h_n)\, dS - \int_{(S)} \rho \mathbf{n} \cdot \mathbf{V}(\mathcal{K} + \mathcal{U}) dS, \tag{53}$$

where

$$h_n = \mathbf{n} \cdot \mathbf{h}. \tag{54}$$

The equation of the energy balance in local form can be written as

$$\rho \left[\frac{d\mathcal{U}}{dt} + \mathbf{V} \cdot \nabla \mathcal{U} \right] = \mathbf{T}^T \cdot\cdot (\nabla \mathbf{V} + \mathbf{E} \times \boldsymbol{\omega}) + \mathbf{M}^T \cdot\cdot \nabla\, \boldsymbol{\omega} + \boldsymbol{\nabla} \cdot \mathbf{h} + \rho q. \tag{55}$$

9 Continuum by Lord Kelvin

Let's accept the next assumptions

$$\mathbf{V} = \mathbf{0}, \quad \mathbf{T} = \mathbf{0} \quad \Rightarrow \quad \rho = \text{const}. \tag{56}$$

In such a case the second law by Euler takes a form

$$\nabla \cdot \mathbf{M} + \rho \mathbf{L} = \rho \dot{\mathbf{K}}_2. \tag{57}$$

The equation of the energy balance can be simplified as well

$$\rho \frac{d\mathcal{U}}{dt} = \mathbf{M}^T \cdot\cdot \nabla \boldsymbol{\omega} + \nabla \cdot \mathbf{h} + \rho q. \tag{58}$$

Let's accept one more restriction

$$\mathbf{M} = \mathbf{H} \times \mathbf{I} \quad \Rightarrow \quad \nabla \cdot \mathbf{M} = \nabla \times \mathbf{H}, \tag{59}$$

where $\mathbf{I}$ is unit tensor. Then equation of the energy balance (58) for the isotermic processes takes a form

$$\rho \frac{d\mathcal{U}}{dt} = -\mathbf{H} \cdot \boldsymbol{\nabla} \times \boldsymbol{\omega}. \tag{60}$$

For the kinetic moment we accept the notation

$$c^{-1}\mathbf{E} = \rho \mathbf{K}_2 = \mathbf{P} \cdot \mathbf{C} \cdot \mathbf{P}^T \cdot \boldsymbol{\omega} + \lambda_0 \dot{\beta}_0 \mathbf{P} \cdot \mathbf{n}, \quad c = \text{const}, \tag{61}$$

where

$$\mathbf{C} = \lambda \mathbf{n} \otimes \mathbf{n} + \mu (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}). \tag{62}$$

Making use of (59) and (61) equation (57) can be replaced by

$$\boldsymbol{\nabla} \times \mathbf{H} + \rho \mathbf{L} = \frac{1}{c} \frac{d}{dt} \mathbf{E}. \tag{63}$$

This equation has a form of the first Maxwell equation. Let the turn-tensor $\mathbf{P}$ be represented in the form

$$\mathbf{P} = \mathbf{Q}(\boldsymbol{\theta}) \cdot \mathbf{Q}(\varphi \mathbf{n}), \quad \boldsymbol{\theta} \cdot \mathbf{n} = 0, \tag{64}$$

where the vector $\boldsymbol{\theta}$ and the angle of own rotation are supposed to be small. In such a case we have

$$\boldsymbol{\omega} = \dot{\boldsymbol{\theta}} + \dot{\varphi} \mathbf{n} = \dot{\boldsymbol{\vartheta}}, \quad \boldsymbol{\vartheta} = \boldsymbol{\theta} + \varphi \mathbf{n}. \tag{65}$$

Equation (60) can be rewritten now as

$$\rho \frac{d\mathcal{U}}{dt} = -\mathbf{H} \cdot \frac{d}{dt} \boldsymbol{\nabla} \times \boldsymbol{\vartheta}. \tag{66}$$

Let's accept the simplest representation for the specific internal energy

$$\rho \mathcal{U} = \frac{1}{2} \kappa |\boldsymbol{\nabla} \times \boldsymbol{\vartheta}|^2, \quad \kappa = \text{const}. \tag{67}$$

Then for the vector **H** we obtain

$$\mathbf{H} = -\kappa \boldsymbol{\nabla} \times \boldsymbol{\vartheta}. \tag{68}$$

If we take into account equality (62) then expression (61) can be rewritten as

$$c^{-1}\mathbf{E} = \mu\dot{\boldsymbol{\theta}} + (\lambda\dot{\varphi} + \lambda_0\dot{\beta})\,\mathbf{n} + \lambda_0\dot{\beta}\boldsymbol{\theta} \times \mathbf{n}. \tag{69}$$

In order to get the simplest case, let's accept the restrictions

$$\lambda = \mu, \quad \lambda_0 = 0. \tag{70}$$

Then we obtain

$$\mathbf{E} = \mu c \frac{d}{dt}\boldsymbol{\vartheta}. \tag{71}$$

From equations (68) and (71) it follows

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{d}{dt}\mathbf{H}. \tag{72}$$

Let's write down equation (63) and (72) together

$$\boldsymbol{\nabla} \times \mathbf{H} + \rho\mathbf{L} = \frac{1}{c}\frac{d}{dt}\mathbf{E}, \quad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{d}{dt}\mathbf{H}, \qquad \kappa = \mu c^2. \tag{73}$$

The obtained equations are the classical equations by Maxwell.

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